Quantum Mechanics
Second Quantization

Second Quantization

- Quantum Many-Body States
- Identical Particles

Second quantization is a formalism used to describe quantum many-body systems of identical particles.

- Classical mechanics: each particle is labeled by a distinct position \( r_i \) ⇒ any different configuration of \( \{r_i\} \) correspond to a different classical many-body state.

- Quantum mechanics: particles are identical, such that exchanging two particles \((r_i \leftrightarrow r_j)\) does not lead to a different quantum many-body state.

Permutation symmetry of identical particles ⇒ joint probability distribution must be invariant under permutation:

\[
p(\ldots, r_i, \ldots, r_j, \ldots) = p(\ldots, r_j, \ldots, r_i, \ldots),
\]

where the probability distribution \( p \) is related to the many-body wave function \( \Psi \) by

\[
p(\ldots, r_i, \ldots, r_j, \ldots) = |\Psi(\ldots, r_i, \ldots, r_j, \ldots)|^2.
\]

The wave function can only change up to an overall phase factor.

\[
\Psi(\ldots, r_i, \ldots, r_j, \ldots) = e^{i\varphi}\Psi(\ldots, r_j, \ldots, r_i, \ldots).
\]

It forms as a one-dimensional representation of the permutation group. Mathematical fact: there are only two 1-dim representations for any permutation group,

- trivial representation ⇒ bosons

\[
\Psi_B(\ldots, r_i, \ldots, r_j, \ldots) = +\Psi_B(\ldots, r_j, \ldots, r_i, \ldots)
\]

- sign representation ⇒ fermions

\[
\Psi_F(\ldots, r_i, \ldots, r_j, \ldots) = -\Psi_F(\ldots, r_j, \ldots, r_i, \ldots)
\]

Dirac Notations

Let us rephrase this using Dirac ket-state notation (more concise). Consider a complete set of single-particle states \( |\alpha\rangle \) (labeled by \( \alpha \))
\[ |\alpha\rangle = \int d^d r \psi_\alpha(r) |r\rangle, \]

where \( \psi_\alpha(r) \) is the wave function representing the state.

- A two-particle state with the 1st particle in \( |\alpha_1\rangle \) and the 2nd particle in \( |\alpha_2\rangle \) will be described by

\[
|\alpha_1\rangle \otimes |\alpha_2\rangle = \int d^d r_1 \int d^d r_2 \psi_{\alpha_1}(r_1) \psi_{\alpha_2}(r_2) |r_1\rangle \otimes |r_2\rangle
= \int d^d r_1 \int d^d r_2 \Psi(r_1, r_2) |r_1\rangle \otimes |r_2\rangle.
\]

\[
\Psi(r_1, r_2) = \psi_{\alpha_1}(r_1) \psi_{\alpha_2}(r_2) \]

is identified as the two-body wave function.

- Exchanging \( r_1 \leftrightarrow r_2 \) in the wave function \( \Psi(r_1, r_2) \) leads to a new wave function \( \Psi'(r_1, r_2) \)

\[
\Psi'(r_1, r_2) = \Psi(r_2, r_1) = \psi_{\alpha_1}(r_2) \psi_{\alpha_2}(r_1) = \psi_{\alpha_2}(r_1) \psi_{\alpha_1}(r_2),
\]

which corresponds to a new state

\[
\int d^d r_1 \int d^d r_2 \psi'(r_1, r_2) |r_1\rangle \otimes |r_2\rangle
= \int d^d r_1 \int d^d r_2 \psi_{\alpha_2}(r_1) \psi_{\alpha_1}(r_2) |r_1\rangle \otimes |r_2\rangle
= |\alpha_2\rangle \otimes |\alpha_1\rangle,
\]

describing a two-particle state with the 1st particle in \( |\alpha_2\rangle \) and the 2nd particle in \( |\alpha_1\rangle \).

**Conclusion**: exchanging the positions of two particles \( (r_1 \leftrightarrow r_2) \) \( \Leftrightarrow \) exchanging the labels of the single-particle state \( (\alpha_1 \leftrightarrow \alpha_2) \).

### First-Quantized States

First-quantization approach:

- Suppose the **single-particle Hilbert space** is \( D \) dimensional, spanned by the **single-particle basis states** \( |\alpha\rangle \) \( (\alpha = 1, 2, \ldots, D) \).

- The **many-body Hilbert space** of \( N \) particles will be \( D^N \) dimensional, spanned by the **many-body basis states**

\[
|\{\alpha\}\rangle \equiv |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots \otimes |\alpha_N\rangle,
\]

where \( \alpha_i = 1, 2, \ldots, D \) labels the state of the \( i \)-th particle.

- A generic **first-quantized state** is a linear superposition of these basis states

\[
|\Psi\rangle = \sum_{[\alpha]} \Psi[\alpha] |[\alpha]\rangle,
\]

where the coefficient \( \Psi[\alpha] \in \mathbb{C} \) is also called the **many-body wave function** (as a more general function of labels \( a_i \); not positions \( r_i \)).

Most of the first-quantized states are **not** qualified to describe systems of **identical particles**.

- For identical **bosons**, \( \Psi[\alpha] \) must be **symmetric**
\[ \Psi_B(\ldots, \alpha_i, \ldots, \alpha_j, \ldots) = + \Psi_B(\ldots, \alpha_j, \ldots, \alpha_i, \ldots) \] 

- For identical fermions, \( \Psi[\alpha] \) must be antisymmetric

\[ \Psi_F(\ldots, \alpha_i, \ldots, \alpha_j, \ldots) = - \Psi_F(\ldots, \alpha_j, \ldots, \alpha_i, \ldots) \]

These states only span a subspace of the first-quantized Hilbert space.

\[ \text{unphysical states} \]

\[ \text{bosonic states} \]

\[ \text{fermionic states} \]

\[ \text{first-quantized states} \]

We would like to pick out (or construct) the basis states for the bosonic and fermionic subspaces. Starting from a generic basis state \( |\alpha \rangle \), we can construct

- **bosonic states by symmetrization**

\[
S |\alpha \rangle = S |\alpha_1 \rangle \otimes |\alpha_2 \rangle \otimes \ldots \otimes |\alpha_N \rangle \\
= \sum_{\pi \in S_N} |\alpha_{\pi(1)} \rangle \otimes |\alpha_{\pi(2)} \rangle \otimes \ldots \otimes |\alpha_{\pi(N)} \rangle, \tag{14}
\]

- **fermionic states by antisymmetrization**

\[
\mathcal{A} |\alpha \rangle = \mathcal{A} |\alpha_1 \rangle \otimes |\alpha_2 \rangle \otimes \ldots \otimes |\alpha_N \rangle \\
= \sum_{\pi \in S_N} (-)^\pi |\alpha_{\pi(1)} \rangle \otimes |\alpha_{\pi(2)} \rangle \otimes \ldots \otimes |\alpha_{\pi(N)} \rangle, \tag{15}
\]

\( \pi \) denotes an \( S_N \) group element and \((-)^\pi\) is the permutation sign of \( \pi \).

\[
(-)^\pi = \begin{cases} 
+1 & \text{if } \pi \text{ has even number of inversions} \\
-1 & \text{if } \pi \text{ has odd number of inversions}
\end{cases} \tag{16}
\]

An inversion is a pair \((x, y)\) such that \( x < y \) and \( \pi(x) > \pi(y) \). Take the \( S_3 \) group for example:

\[
\pi(123) \quad 123 \quad 231 \quad 312 \quad 321 \quad 213 \quad 132 \\
(-)^\pi \quad +1 \quad +1 \quad -1 \quad -1 \quad -1 \quad -1 \tag{17}
\]

- **Examples of bosonic states** (unnormalized):

\[
S |\alpha \rangle \otimes |\beta \rangle = |\alpha \rangle \otimes |\beta \rangle + |\beta \rangle \otimes |\alpha \rangle, \quad \text{(assuming } \alpha \neq \beta) \\
S |\alpha \rangle \otimes |\alpha \rangle = |\alpha \rangle \otimes |\alpha \rangle. \tag{18}
\]

- **Examples of fermionic states** (unnormalized):

\[
\mathcal{A} |\alpha \rangle \otimes |\beta \rangle = |\alpha \rangle \otimes |\beta \rangle - |\beta \rangle \otimes |\alpha \rangle, \quad \text{(assuming } \alpha \neq \beta) \\
\mathcal{A} |\alpha \rangle \otimes |\alpha \rangle = 0 \Rightarrow \text{no such fermionic state.} \tag{19}
\]

**Pauli exclusion principle**: two (or more) identical fermions can not occupy the same state simultaneously.
Originally $|\alpha\rangle \otimes |\beta\rangle$ and $|\beta\rangle \otimes |\alpha\rangle$ (for $\alpha \neq \beta$) are two orthogonal first-quantized states, under either symmetrization or antisymmetrization, they correspond to the same state (up to $\pm 1$ overall phase)

$$S |\alpha\rangle \otimes |\beta\rangle = S |\beta\rangle \otimes |\alpha\rangle, \quad A |\alpha\rangle \otimes |\beta\rangle = -A |\beta\rangle \otimes |\alpha\rangle. \quad (20)$$

- The first-quantized Hilbert space is redundant $\Rightarrow$ there are fewer basis states in the bosonic and fermionic subspaces.
- Consider $N$ particles, each can take one of $D$ different single-particle states,
  - the dimension of bosonic subspace:
    $$D_B = \frac{(N+D-1)!}{N! (D-1)!}. \quad (21)$$
  - the dimension of fermionic subspace:
    $$D_F = \frac{D!}{N! (D-N)!}. \quad (22)$$

It turns out that $D_B + D_F \leq D^N$ as long as $N > 1 \Rightarrow$ the remaining basis states in the first-quantized Hilbert space are unphysical (for identical particles).

These unphysical states are annoying; we can not combine the states in the Hilbert space freely. We must always remember to symmetrized/antisymmetrized the state. $\Rightarrow$ Is there a better way to organize the many-body Hilbert space, such that all states in the space are physical?

- **Second-Quantized States (Fock States)**

  Sometimes difficulties in physics arise from the inappropriate language we used. There are two different ways to describe many-body states:
  - In **first-quantization**, we ask: Which particle is in which state?
  - In **second-quantization**, we ask: How many particles are there in every state?

The question we ask in first-quantization is inappropriate: if the particles are identical, it will be impossible to tell which particle is which in the first place. We need to switch to a new language

| $|\alpha\rangle \otimes |\beta\rangle$ | $|\beta\rangle \otimes |\alpha\rangle$ |
|---------------------------------|---------------------------------|
| the 1st particle on $|\alpha\rangle$ | the 1st particle on $|\beta\rangle$ |
| the 2nd particle in $|\beta\rangle$ | the 2nd particle in $|\alpha\rangle$ |

there is one particle in $|\alpha\rangle$, another particle in $|\beta\rangle$.

The new description does not require the labeling of particles. $\Rightarrow$ It contains no redundant information. $\Rightarrow$ It leads to a more precise and succinct description.

In the second-quantization approach,
- Each **basis state** in the many-body Hilbert space is labeled by a set of **occupation numbers** $n_\alpha$ (for $\alpha = 1, 2, \ldots, D$)
meaning that there are \(n_\alpha\) particles in the state \(|\alpha\rangle\).

\[
\begin{align*}
n_\alpha &= \begin{cases} 
  0, 1, 2, 3, \ldots & \text{bosons}, \\
  0, 1 & \text{fermions}.
\end{cases}
\end{align*}
\]

- For **bosons**, \(n_\alpha\) can be any non-negative integer.
- For **fermions**, \(n_\alpha\) can only take 0 or 1, due to the Pauli exclusion principle.
- The occupation numbers \(n_\alpha\) sum up to the total number of particles, i.e. \(\sum \alpha n_\alpha = N\).
- The states \(|[n]\rangle\) are also known as **Fock states**.
- All Fock states form a complete set of basis for the many-body Hilbert space, or the **Fock space**.
- Any generic **second-quantized** many-body state is a linear combination of Fock states,

\[
|\Psi\rangle = \sum_{[n]} \Psi[n] |[n]\rangle.
\]

### Representation of Fock States

The **first-** and the **second-quantization** formalisms can both provide legitimate description of identical particles. (The first-quantization is just awkward to use, but it is still valid.)

**Every Fock state has a first-quantized representation.**

- The Fock state with all occupation numbers to be zero is called the **vacuum state**, denoted as

\[
|0\rangle \equiv |\ldots, 0, \ldots\rangle
\]

It corresponds to the **tensor product unit** in the first-quantization, which can be written as

\[
|0\rangle_F = |0\rangle_B = \mathbf{1}.
\]

We use a subscript \(B/F\) to indicate whether the Fock state is **bosonic** \((B)\) or **fermionic** \((F)\). For vacuum state, there is no difference between them.

- The Fock state with only one non-zero occupation number is a **single-mode Fock state**, denoted as

\[
|n_\alpha\rangle = |\ldots, 0, n_\alpha, 0, \ldots\rangle
\]

In terms of the first-quantized states

\[
\begin{align*}
|1_\alpha\rangle_B &= |1_\alpha\rangle_F = |\alpha\rangle, \\
|2_\alpha\rangle_B &= |\alpha\rangle \otimes |\alpha\rangle, \\
|3_\alpha\rangle_B &= |\alpha\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle, \\
|n_\alpha\rangle_B &= |\alpha\rangle \otimes |\alpha\rangle \otimes \ldots \otimes |\alpha\rangle \equiv |\alpha\rangle \otimes^{n_\alpha}.
\end{align*}
\]
• For **multi-mode Fock states** (meaning more than one single-particle state \(|\alpha\rangle\) is involved), the first-quantized state will involve appropriate symmetrization depending on the particle statistics. For example,

\[
|1_\alpha, 1_\beta\rangle_B = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\beta\rangle + |\beta\rangle \otimes |\alpha\rangle),
\]

\[
|1_\alpha, 1_\beta\rangle_F = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\beta\rangle - |\beta\rangle \otimes |\alpha\rangle).
\]

Note the difference between bosonic and fermionic Fock states (even if their occupation numbers are the same). Here are more examples

\[
|2_\alpha, 1_\beta\rangle_B = \frac{1}{\sqrt{3}} (|\alpha\rangle \otimes |\alpha\rangle \otimes |\beta\rangle + |\alpha\rangle \otimes |\beta\rangle \otimes |\alpha\rangle + |\beta\rangle \otimes |\alpha\rangle \otimes |\alpha\rangle),
\]

\[
|1_\alpha, 1_\beta, 1_\gamma\rangle_F = \frac{1}{\sqrt{6}} (|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle + |\beta\rangle \otimes |\gamma\rangle \otimes |\alpha\rangle + |\gamma\rangle \otimes |\alpha\rangle \otimes |\beta\rangle - |\gamma\rangle \otimes |\beta\rangle \otimes |\alpha\rangle - |\beta\rangle \otimes |\alpha\rangle \otimes |\gamma\rangle - |\alpha\rangle \otimes |\gamma\rangle \otimes |\beta\rangle).
\]

Ok, you get the idea. In general, the Fock state can be represented as

• for **bosons**, \(\mathcal{S}\)

\[
[[n]]_B = \left( \prod_a \frac{n_a!}{N!} \right)^{1/2} \mathcal{S} \otimes |\alpha\rangle^{\otimes n_a}.
\]

• for **fermions**, \(\mathcal{A}\)

\[
[[n]]_F = \frac{1}{\sqrt{N!}} \mathcal{A} \otimes |\alpha\rangle^{\otimes n_a}.
\]

\(\mathcal{S}\) and \(\mathcal{A}\) are symmetrization and antisymmetrization operators defined in Eq. (14) and Eq. (15).

### Creation and Annihilation Operators

### State Insertion and Deletion

The **creation** and **annihilation operators** are introduced to *create* and *annihilate* particles in the quantum many-body system, as indicated by their names. The first step towards defining them is to understand how to *insert* and *delete* a single-particle state from the first-quantized state in a **symmetric** (or **antisymmetric**) manner.

Let us first declare some notations:

• Let \(|\alpha\rangle\), \(|\beta\rangle\) be single-particle states.

• Let \(I\) be the tensor identity (meaning that \(|\alpha\rangle \otimes I = I \otimes |\alpha\rangle = |\alpha\rangle\)).

• Let \(|\Psi\rangle\), \(|\Phi\rangle\) be generic **first-quantized states** as in Eq. (11).
Now we define the **insertion operator** $\triangleright_\pm$ and **deletion operator** $\triangleleft_\pm$ by the following rules:

- **Linearity** (for $a, b \in \mathbb{C}$)
  \[
  |a\rangle \triangleright_\pm (a|\Psi\rangle + b|\Phi\rangle) = a|a\rangle \triangleright_\pm |\Psi\rangle + b|a\rangle \triangleright_\pm |\Phi\rangle,
  \]
  \[
  |a\rangle \triangleleft_\pm (a|\Psi\rangle + b|\Phi\rangle) = a|a\rangle \triangleleft_\pm |\Psi\rangle + b|a\rangle \triangleleft_\pm |\Phi\rangle.
  \]

- **Vacuum action**
  \[
  |a\rangle \triangleright_\pm 1 = |a\rangle,
  \]
  \[
  |a\rangle \triangleleft_\pm 1 = 0.
  \]

- **Recursive relation**
  \[
  |a\rangle \triangleright_\pm |\beta\rangle \otimes |\Psi\rangle = |a\rangle \otimes |\beta\rangle \otimes |\Psi\rangle \pm |\beta\rangle \otimes (|a\rangle \triangleright_\pm |\Psi\rangle),
  \]
  \[
  |a\rangle \triangleleft_\pm |\beta\rangle \otimes |\Psi\rangle = \langle a \ | \beta \rangle |\Psi\rangle \pm |\beta\rangle \otimes (|a\rangle \triangleleft_\pm |\Psi\rangle).
  \]

(\[34\])

$\langle a \ | \beta \rangle = \delta_{ab}$ if $|a\rangle$ and $|\beta\rangle$ are orthonormal basis states. The subscript $\pm$ of the insertion or deletion operators indicates whether symmetrization ($+$) or antisymmetrization ($-$) is implemented.

### Boson Creation and Annihilation

- The **boson creation operator** $b_a^\dagger$ adds a boson to the single-particle state $|a\rangle$, **increasing** the occupation number by one $n_a \to n_a + 1$. It acts on a $N$-particle first-quantized state $|\Psi\rangle$ as
  \[
  b_a^\dagger |\Psi\rangle = \frac{1}{\sqrt{N+1}} |a\rangle \triangleright_+ |\Psi\rangle,
  \]
  where $|a\rangle \triangleright_+ $ **inserts** the single-particle state $|a\rangle$ to $N+1$ possible insertion positions **symmetrically**.

- The **boson annihilation operator** $b_a$ removes a boson from the single-particle state $|a\rangle$, **reducing** the occupation number by one $n_a \to n_a - 1$ (while $n_a > 0$). It acts on a $N$-particle first-quantized state $|\Psi\rangle$ as
  \[
  b_a |\Psi\rangle = \frac{1}{\sqrt{N}} |a\rangle \triangleleft_+ |\Psi\rangle,
  \]
  where $|a\rangle \triangleleft_+ $ **removes** the single-particle state $|a\rangle$ from $N$ possible deletion positions **symmetrically**.

### Single-Mode Fock States

Based on these definitions, we can show that the creation and annihilation operators acting on single-mode Fock states as
\[
|a\rangle \triangleright_+ |n_a\rangle = \frac{1}{\sqrt{n_a+1}} |a\rangle \triangleright_+ |a\rangle \otimes^{n_a}\]
\[
\begin{align*}
\frac{n_\alpha + 1}{\sqrt{n_\alpha + 1}} |\alpha\rangle \otimes (n_\alpha + 1) \\
= \sqrt{n_\alpha + 1} |n_\alpha + 1\rangle.
\end{align*}
\]

Thus we conclude

\[
\begin{align*}
b_{\alpha}^\dagger |n_\alpha\rangle &= \sqrt{n_\alpha + 1} |n_\alpha + 1\rangle, \\
b_{\alpha} |n_\alpha\rangle &= \sqrt{n_\alpha} |n_\alpha - 1\rangle.
\end{align*}
\]  

(40)

- Especially, when acting on the vacuum state

\[
\begin{align*}
b_{\alpha}^\dagger |0_\alpha\rangle &= |1_\alpha\rangle, \\
b_{\alpha} |0_\alpha\rangle &= 0.
\end{align*}
\]  

(41)

- Using Eq. (41), we can show that

\[
\begin{align*}
b_{\alpha}^\dagger b_{\alpha} |n_\alpha\rangle &= n_\alpha |n_\alpha\rangle,
\end{align*}
\]  

(42)

meaning that \( b_{\alpha}^\dagger b_{\alpha} \) is the **boson number operator** of the |\alpha\rangle state.

All the single-mode Fock state can be constructed by the boson creation operator from the vacuum state

\[
|n_\alpha\rangle = \frac{1}{\sqrt{n_\alpha!}} (b_{\alpha}^\dagger)^{n_\alpha} |0_\alpha\rangle.
\]  

(43)

- **Generic Fock States**

  The above result can be generalized to any Fock state of bosons

\[
\begin{align*}
b_{\alpha}^\dagger \ldots, n_\beta, n_\gamma, \ldots B &= \sqrt{n_\alpha + 1} \ldots, n_\beta, n_\alpha + 1, n_\gamma, \ldots B, \\
b_{\alpha} \ldots, n_\beta, n_\gamma, \ldots B &= \sqrt{n_\alpha} \ldots, n_\beta, n_\alpha - 1, n_\gamma, \ldots B.
\end{align*}
\]  

(44)

These two equations can be considered as the **defining properties** of boson creation and annihilation operators.

- **Operator Identities**

  Eq. (45) implies the following operator identities
These relations can be considered as the algebraic definition of boson creation and annihilation operators.

**Fermion Creation and Annihilation**

- The fermion creation operator \( c_\alpha^\dagger \) adds a fermion to the single-particle state \(|\alpha\rangle\), increasing the occupation number by one \( n_\alpha \to n_\alpha + 1 \) (while \( n_\alpha = 0 \)). It acts on a \( N \)-particle first-quantized state \(|\Psi\rangle\) as

\[
c_\alpha^\dagger |\Psi\rangle = \frac{1}{\sqrt{N+1}} |\alpha\rangle \bowtie_{-} |\Psi\rangle,
\]

where \(|\alpha\rangle \bowtie_{-} \) inserts the single-particle state \(|\alpha\rangle\) to \( N + 1 \) possible insertion positions anti-symmetrically.

- The fermion annihilation operator \( c_\alpha \) removes a fermion from the single-particle state \(|\alpha\rangle\), reducing the occupation number by one \( n_\alpha \to n_\alpha - 1 \) (while \( n_\alpha = 1 \)). It acts on a \( N \)-particle first-quantized state \(|\Psi\rangle\) as

\[
c_\alpha |\Psi\rangle = \frac{1}{\sqrt{N}} |\alpha\rangle \bowtie_{-} |\Psi\rangle,
\]

where \(|\alpha\rangle \bowtie_{-} \) removes the single-particle state \(|\alpha\rangle\) from \( N \) possible deletion positions anti-symmetrically.

**Single-Mode Fock States**

Based on these definitions, we can show that the creation and annihilation operators acting on single-mode Fock states as

\[
c_\alpha^\dagger |0_\alpha\rangle = |\alpha\rangle \bowtie_{-} 1 = |\alpha\rangle = |1_\alpha\rangle
\]

\[
c_\alpha^\dagger |1_\alpha\rangle = \frac{1}{\sqrt{2}} |\alpha\rangle \bowtie_{-} |\alpha\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle \otimes |\alpha\rangle - |\alpha\rangle \otimes |\alpha\rangle) = 0
\]

\[
c_\alpha |0_\alpha\rangle = 0
\]

\[
c_\alpha |1_\alpha\rangle = |\alpha\rangle \bowtie_{-} |\alpha\rangle = 1 = |0_\alpha\rangle.
\]

Thus we conclude (note that \( n_\alpha = 0, 1 \) only take two values)

\[
c_\alpha^\dagger |n_\alpha\rangle = \sqrt{1 - n_\alpha} |1 - n_\alpha\rangle,
\]

\[
c_\alpha |n_\alpha\rangle = \sqrt{n_\alpha} |1 - n_\alpha\rangle.
\]
meaning that $c^\dagger_\alpha c_\alpha$ is the fermion number operator of the $|\alpha\rangle$ state.

All the single-mode Fock state can be constructed by the boson creation operator from the vacuum state

$$|n_\alpha\rangle = (c^\dagger_\alpha)^{n_\alpha}|0_\alpha\rangle.$$  (53)

**Generic Fock States**

The above result can be generalized to any Fock state of bosons

$$c^\dagger_\alpha|\cdots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle_F = (-)^{\sum n_\beta} \sqrt{1 - n_\alpha}|\cdots, n_\beta, 1 - n_\alpha, n_\gamma, \ldots\rangle_F,$$

$$c_\alpha|\cdots, n_\beta, n_\alpha, n_\gamma, \ldots\rangle_F = (-)^{\sum n_\beta} \sqrt{n_\alpha}|\cdots, n_\beta, 1 - n_\alpha, n_\gamma, \ldots\rangle_F.$$  (54)

These two equations can be considered as the defining properties of fermion creation and annihilation operators.

**Operator Identities**

Eq. (54) implies the following operator identities

$$\{c^\dagger_\alpha, c^\dagger_\beta\} = \{c_\alpha, c_\beta\} = 0, \quad \{c_\alpha, c^\dagger_\beta\} = \delta_{\alpha\beta}.$$  (55)

These relations can be considered as the algebraic definition of fermion creation and annihilation operators.