PHYS 212B(Winter 2020) Final(2020 March) Deadline: March 19, 2020

Name: PID (optional):

Please finish all three problems as much as you can. Submit your answer before 23:59 (midnight), March 19, 2020, by emailing to the TA Dachuan Lu ldachuan@ucsd.edu and Prof. You yzyou@ucsd.edu. If you have any confusion about the problem, please also contact the TA or the professor by email.

1. Schwinger boson representation

Two-dimensional quantum harmonic oscillator hosts two decoupled free bosons, whose annihilation operators can be denoted as a and b respectively

$$a = \frac{x + \mathrm{i}p_x}{\sqrt{2}}, \quad b = \frac{y + \mathrm{i}p_y}{\sqrt{2}}.$$

They satisfy the commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$ and $[a, b] = [a, b^\dagger] = 0$. The system enjoys the U(2) symmetry, which contains a SU(2) subgroup. Let us explore how to establish the SU(2) representations using boson operators.

Define $S^x = \frac{1}{2}(a^\dagger b + b^\dagger a)$, $S^z = \frac{1}{2}(a^\dagger a - b^\dagger b)$.

(a) Find S^y in terms of a, b. [Hint: such that $\boldsymbol{S} \times \boldsymbol{S} = \mathrm{i}\boldsymbol{S}$]

(b) Show that S^y is actually related to the angular momentum operator $L = xp_y - yp_x$ of the harmonic oscillator as $S^y = \frac{1}{2}L$.

(c) Define the following set of states with $s = 0, 1/2, 1, \cdots$ and $m = -s, -s+1, \cdots, s-1, s$ (they are called the Schwinger boson representation),

$$|s, m\rangle = \frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}} \frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}} |\Omega\rangle,$$

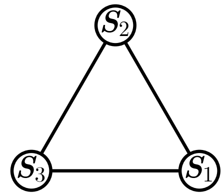
where $|\Omega\rangle$ is the state annihilated by a, b, i.e. $a|\Omega\rangle = b|\Omega\rangle = 0$. Show that the state $|s, m\rangle$ is indeed a common eigenstate of $\boldsymbol{S}^2 = (S^x)^2 + (S^y)^2 + (S^z)^2$ and S^z with eigenvalues $s(s+1)$ and m respectively. [Hint: using particle number basis.]

2. Spin-$\frac{1}{2}$ on a triangle

Consider spin-$\frac{1}{2}$ on each site of a triangle. The Hilbert space is spanned by the following $2^3 = 8$ basis states: $\{|\uparrow\uparrow\uparrow\rangle, |\uparrow\uparrow\downarrow\rangle, |\uparrow\downarrow\uparrow\rangle, |\uparrow\downarrow\downarrow\rangle, |\downarrow\uparrow\uparrow\rangle, |\downarrow\uparrow\downarrow\rangle, |\downarrow\downarrow\uparrow\rangle, |\downarrow\downarrow\downarrow\rangle\}$. We will call this basis the *tensor product basis*. The Hamiltonian for this system is

$$H = \mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1,$$

where $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$ and $S_i^a = \frac{1}{2}\sigma_i^a$, where $i = 1, 2, 3$ labels the triangle sites.



(a) Write H in terms of the total spin operator $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3$ (express H as a polynomial of $\boldsymbol{S}$). Using the fact that three spin-1/2's fuse to two spin-1/2's and a spin-3/2 as $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \to \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$, determine the eigenvalues of H and the corresponding degeneracies.

(b) Let $|s, m\rangle$ be the common eigenstate of the total spin $\boldsymbol{S}^2$ and its z-component $S^z = \sum_{i=1}^3 S_i^z$ with eigenvalues $s(s+1)$ and m respectively. Verify that $|\frac{3}{2}, +\frac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle$ (by showing that $|\uparrow\uparrow\uparrow\rangle$ in deed has the desired eigenvalues).

(c) We can use the spin raising and lowering operator $S^{\pm} = S^x \pm \mathrm{i} S^y$ to construct the represent of $|s, m\rangle$ state in the *tensor product basis* (i.e. $\{|\uparrow\rangle, |\downarrow\rangle\}^{\otimes 3}$). Starting with $|\frac{3}{2}, +\frac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle$, by applying the spin lowering operator on both sides, find the expression for other states $|s = \frac{3}{2}, m\rangle$ in the spin-3/2 sector (please provide the answer for $m = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}$).

3. Quantum rotor under perturbation

Consider a quantum rotor described by the Hamiltonian

$$H = \frac{1}{2}\Big(\hat{N} - \frac{1}{2}\Big)^2 - g\cos(2\theta),$$

where $\hat{N} = -\mathrm{i}\partial_\theta$ is the angular momentum operator and $g\cos(2\theta)$ is a small potential that can be treated as perturbation. Let $|N\rangle$ be the eigenstate of $\hat{N}$ with eigenvalue N (i.e. $\hat{N}|N\rangle = N|N\rangle$).

(a) For the unperturbed Hamiltonian $H_0 = \frac{1}{2}\big(\hat{N} - \frac{1}{2}\big)^2$, show that the angular momentum eigenstates $|N\rangle$ are also energy eigenstates, what are their corresponding energies? Show that every level is two-fold degenerated (by showing that which and which states are degenerated in energy).

(b) Represent the perturbation $V = -g\cos(2\theta)$ in the $|N\rangle$ basis, show that the perturbation never connects degenerated levels. Therefore we can still proceed with non-degenerate perturbation theory, even though the unperturbed Hamiltonian H_0 has degenerated levels.

(c) Calculate the perturbative correction of every energy level E_N (as labeled by N) to the second order in g. Show that the two-fold degeneracies are still not lifted for all levels.