PHYS 130C

Part 3: Quantum Optics

Quantization of Light

■ Classical Electromagnetic Wave

■ Lagrangian Description

Lagrangian density for *free* **electromagnetic field**:

$$\boxed{\mathcal{L} = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right),} \tag{1}$$

- Electromagnetic field is the *physical observable*:
  - $\boldsymbol{E}$ - **electric field**,
  - $\boldsymbol{B}$ - **magnetic field**,

$$\boxed{\begin{aligned}\boldsymbol{E} &= -\nabla\Phi - \partial_t \boldsymbol{A},\\ \boldsymbol{B} &= \nabla \times \boldsymbol{A}.\end{aligned}} \tag{2}$$

  - Note: the **speed of light** $c = 1$ is set to unity.
- **Gauge field**: $(\Phi, \boldsymbol{A})$ as *generalized coordinates* (state variables)
  - Φ - **scalar potential**,
  - $\boldsymbol{A}$ - **vector potential**.

■ Maxwell Equations

The **Maxwell equations** describes the motion of electromagnetic field. In the *free space* (without sources), they are

$$\boxed{\begin{aligned}&\nabla \times \boldsymbol{E} + \partial_t \boldsymbol{B} = 0,\\ &\nabla \cdot \boldsymbol{B} = 0,\\ &\nabla \cdot \boldsymbol{E} = 0,\\ &\nabla \times \boldsymbol{B} - \partial_t \boldsymbol{E} = 0.\end{aligned}} \tag{3}$$

- The first two equations follows from Eq. (2), by definition.

Exc 1 Verify that Eq. (2) implies the first two equations in Eq. (3).

- The last two equations follows from the variational principle $\delta\mathcal{L} = 0$,

$$\begin{aligned} &\frac{\delta\mathcal{L}}{\delta\Phi} = 0 \Rightarrow \nabla\cdot\boldsymbol{E} = 0,\\ &\frac{\delta\mathcal{L}}{\delta\boldsymbol{A}} = 0 \Rightarrow \nabla\times\boldsymbol{B} - \partial_t\boldsymbol{E} = 0. \end{aligned} \tag{4}$$

Exc 2 Drive Eq. (4).

Gauge Structure and Gauge Fixing

Gauge structure: physical observables $\boldsymbol{E}$ and $\boldsymbol{B}$ are *invariant* under the following *gauge transformations* induced by any scalar field θ,

$$\boxed{\begin{aligned} &\boldsymbol{A} \to \boldsymbol{A} + \nabla\theta,\\ &\Phi \to \Phi - \partial_t\theta. \end{aligned}} \tag{5}$$

Exc 3 Show that the gauge transformation Eq. (5) leaves Eq. (2) invariant.

- Gauge structure is a **redundancy** in the gauge theory: there are multiple *state variables* (gauge field Φ, $\boldsymbol{A}$) encoding the same *physical observables* (electromagnetic field $\boldsymbol{E}$, $\boldsymbol{B}$).
- **Gauge fixing** is a procedure to eliminate the gauge redundancy, by using *gauge transformation* to (partially) fix the gauge field configuration.

The **Coulomb gauge** is one commonly used gauge choice:

$$\boxed{\begin{aligned} &\Phi = 0,\\ &\nabla\cdot\boldsymbol{A} = 0. \end{aligned}} \tag{6}$$

Gauge fixing procedure:

- Freedom to use: θ field (through out the spacetime).
- If $\Phi \neq 0$, use $\Phi \to \Phi - \partial_t\theta$ to fix $\Phi = 0$, by setting

$$\theta = \int_0^t dt\,\Phi + \theta_{t=0}, \tag{7}$$

where $\theta_{t=0}$ is still free to tune through out the space.

- With $\Phi = 0$ fixed, $\boldsymbol{E} = -\partial_t\boldsymbol{A}$, then the Coulomb law implies

$$\nabla\cdot\boldsymbol{E} = 0 \Rightarrow -\partial_t\nabla\cdot\boldsymbol{A} = 0 \Rightarrow \nabla\cdot\boldsymbol{A} = \nabla\cdot\boldsymbol{A}_{t=0}. \tag{8}$$

- At the $t = 0$ time slice, if $\nabla \cdot \boldsymbol{A}_{t=0} \neq 0$, use $\boldsymbol{A}_{t=0} \to \boldsymbol{A}_{t=0} + \nabla \theta_{t=0}$ to fix $\nabla \cdot \boldsymbol{A}_{t=0} = 0$, by solving for $\theta_{t=0}$ from

$$\nabla \cdot \boldsymbol{A}_{t=0} + \nabla^2 \theta_{t=0} = 0. \tag{9}$$

Then $\nabla \cdot \boldsymbol{A} = 0$ is also fixed.

[Almost all freedom of θ has been used, only the global shift of θ is still free, which corresponds to a global U(1) symmetry associated with electric charge conservation.]

Under *Coulomb gauge*, $\Phi = 0$ is fixed, $\boldsymbol{A}$ remains as the **generalized coordinate** (with the constraint $\nabla \cdot \boldsymbol{A} = 0$), the conjugate **generalized momentum** is

$$\boxed{\frac{\partial \mathcal{L}}{\partial(\partial_t \boldsymbol{A})} = -\boldsymbol{E}.} \tag{10}$$

Hamiltonian Description

Hamiltonian density for *free* **electromagnetic field**:

$$\boxed{\mathcal{H} = \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right),} \tag{11}$$

Exc 4 | Derive the Hamiltonian density Eq. (11) from the Lagrangian density Eq. (1).

which might as well be written in terms of the generalized coordinate $\boldsymbol{A}$ and the generalized momentum $-\boldsymbol{E}$ as

$$\mathcal{H} = \frac{1}{2}\left(\boldsymbol{E}^2 + (\nabla \times \boldsymbol{A})^2\right). \tag{12}$$

- Hamiltonian dynamics

$$\begin{aligned} \partial_t \boldsymbol{A} &= -\frac{\partial \mathcal{H}}{\partial \boldsymbol{E}} \Rightarrow \partial_t \boldsymbol{A} = -\boldsymbol{E}, \\ \partial_t \boldsymbol{E} &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{A}} \Rightarrow \partial_t \boldsymbol{E} = -\nabla \times (\nabla \times \boldsymbol{A}) = \nabla^2 \boldsymbol{A} - \nabla(\nabla \cdot \boldsymbol{A}). \end{aligned} \tag{13}$$

Use the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, and combine the equations of motion, the vector potential satisfy a **wave equation**,

$$\boxed{\partial_t^2 \boldsymbol{A} - \nabla^2 \boldsymbol{A} = 0.} \tag{14}$$

Electromagnetic Wave

The solution of Eq. (14) describes the **electromagnetic wave** in the free space,

$$\boldsymbol{A}(\boldsymbol{r},\, t) = \sum_{\boldsymbol{k}} \boldsymbol{A}_{\boldsymbol{k}}\, e^{-i\,\omega_k\, t + i\, \boldsymbol{k}\cdot\boldsymbol{r}}. \tag{15}$$

- The *angular frequency* $\omega_{\boldsymbol{k}}$ must satisfy the **dispersion relation**

$$\omega_{\boldsymbol{k}} = |\boldsymbol{k}|, \tag{16}$$

where the speed of light has been set to $c = 1$, compared to the general form of $\omega_{\boldsymbol{k}} = c\,|\boldsymbol{k}|$.

Exc 5 Verify that Eq. (15) is a general solution of Eq. (14), given Eq. (16).

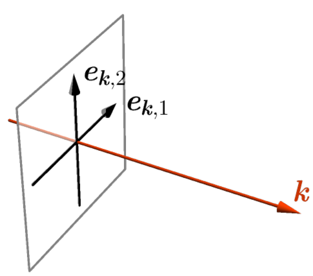
- $\boldsymbol{A}_{\boldsymbol{k}}$ is the *wave amplitudes* (i.e. the Fourier components of $\boldsymbol{A}$, as a complex vector) at each *wave vector* $\boldsymbol{k}$. The gauge constraint $\nabla \cdot \boldsymbol{A} = 0$ further requires

$$\boldsymbol{k} \cdot \boldsymbol{A}_{\boldsymbol{k}} = 0, \tag{17}$$

meaning that the electromagnetic wave is *transverse*. For any $\boldsymbol{k}$, there are only two transverse directions, hence, two independent **polarization directions**, labeled by unit vectors $\boldsymbol{e}_{\boldsymbol{k},\alpha}$ ($\alpha = 1, 2$), such that

$$\boldsymbol{A}_{\boldsymbol{k}} = A_{\boldsymbol{k},1}\, \boldsymbol{e}_{\boldsymbol{k},1} + A_{\boldsymbol{k},2}\, \boldsymbol{e}_{\boldsymbol{k},2} = \sum_{\alpha=1,2} A_{\boldsymbol{k},\alpha}\, \boldsymbol{e}_{\boldsymbol{k},\alpha}, \tag{18}$$

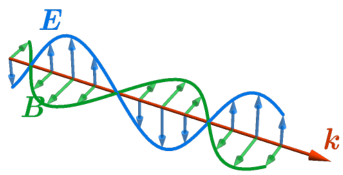
where $A_{\boldsymbol{k},\alpha}$ is the wave amplitude of polarization α with wave vector $\boldsymbol{k}$.



- The corresponding solution of **electromagnetic field** follows from $\boldsymbol{E} = -\partial_t \boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},\, t) &= i \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}}\, \boldsymbol{A}_{\boldsymbol{k}}\, e^{-i\,\omega_k\, t + i\, \boldsymbol{k}\cdot\boldsymbol{r}}, \\ \boldsymbol{B}(\boldsymbol{r},\, t) &= i \sum_{\boldsymbol{k}} \boldsymbol{k} \times \boldsymbol{A}_{\boldsymbol{k}}\, e^{-i\,\omega_k\, t + i\, \boldsymbol{k}\cdot\boldsymbol{r}}. \end{aligned} \tag{19}$$

Here is an illustration of linearly polarized electromagnetic wave.



Quantization of Electromagnetic Field

Canonical Quantization (Real Space)

Canonical quantization is a procedure to transition from *classical mechanics* to *quantum mechanics*. It is based on the principle of promoting *classical observables* (like position and momentum) to *operators* acting on a Hilbert space.

General Procedure:

- Identify the classical *phase space*: a classical system described by generalized *coordinates* q_i and their conjugate *momenta* $p_i := \partial L / \partial \dot{q}_i$.
- Promote *classical variables* to **quantum operators**:

$$\begin{aligned} &q_i \to \hat{q}_i,\ \ p_i \to \hat{p}_i, \\ &H \to \hat{H} = H(\hat{q}_i, \hat{p}_i). \end{aligned} \tag{20}$$

- Impose **canonical commutation relations** between conjugate pairs of coordinates and momenta (setting $\hbar = 1$):

$$\begin{aligned} &\left[\hat{q}_i, \hat{q}_j\right] = \left[\hat{p}_i, \hat{p}_j\right] = 0, \\ &\left[\hat{q}_i, \hat{p}_j\right] = i\, \delta_{ij}\, \mathbb{1}. \end{aligned} \tag{21}$$

(For simplicity, we will omit the operator symbol $\hat{\square}$ in the following, with the understanding that any classical variable in quantum mechanics is promoted to an operator.)

Apply to electromagnetic field. Given that $\boldsymbol{A}$ and $-\boldsymbol{E}$ are generalized *coordinates* and *momenta* [recall Eq. (10)], their *canonical commutation relations* reads

$$\boxed{\begin{aligned} &[A_i(\boldsymbol{r}), A_j(\boldsymbol{r}')] = [-E_i(\boldsymbol{r}), -E_j(\boldsymbol{r}')] = 0, \\ &[A_i(\boldsymbol{r}), -E_j(\boldsymbol{r}')] = i\, \delta_{ij}\, \delta(\boldsymbol{r} - \boldsymbol{r}')\, \mathbb{1}. \end{aligned}} \tag{22}$$

- **Field Operators**: define the following (vectorial) operators

$$\boldsymbol{A}(\boldsymbol{r}) = (A_x(\boldsymbol{r}),\, A_y(\boldsymbol{r}),\, A_z(\boldsymbol{r})),$$
$$\boldsymbol{E}(\boldsymbol{r}) = (E_x(\boldsymbol{r}),\, E_y(\boldsymbol{r}),\, E_z(\boldsymbol{r})), \tag{23}$$

at each point $\boldsymbol{r}$ in the space.

  - Each component is a *Hermitian* operator (corresponding to a *real* variable in the classical limit)

$$A_i^\dagger(\boldsymbol{r}) = A_i(\boldsymbol{r}),$$
$$E_i^\dagger(\boldsymbol{r}) = E_i(\boldsymbol{r}). \tag{24}$$

- In general, $\boldsymbol{A}$ and $\boldsymbol{E}$ are non-commuting operators. They only commute (become independent) if they are
  - at different spacial positions,
  - or along perpendicular directions.

▪ Canonical Quantization (Momentum Space)

Fourier transformation allows us to express *field operators* in the *momentum space*, rather than in *real space*, which can simplify calculations.

- Forward transformation:

$$\boldsymbol{A}_{\boldsymbol{k}} = \int d^3\boldsymbol{r}\, \boldsymbol{A}(\boldsymbol{r})\, e^{-i\, \boldsymbol{k}\cdot\boldsymbol{r}},$$
$$\boldsymbol{E}_{\boldsymbol{k}} = \int d^3\boldsymbol{r}\, \boldsymbol{E}(\boldsymbol{r})\, e^{-i\, \boldsymbol{k}\cdot\boldsymbol{r}}. \tag{25}$$

- Backward transformation:

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \boldsymbol{A}_{\boldsymbol{k}}\, e^{i\, \boldsymbol{k}\cdot\boldsymbol{r}},$$
$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \boldsymbol{E}_{\boldsymbol{k}}\, e^{i\, \boldsymbol{k}\cdot\boldsymbol{r}}. \tag{26}$$

Note: $\sum_{\boldsymbol{k}} := (2\pi)^{-3} \int d^3\boldsymbol{k}$ to properly normalize.

The Fourier components $\boldsymbol{A}_{\boldsymbol{k}}$ and $\boldsymbol{E}_{\boldsymbol{k}}$ are also *operators*, constructed as *linear combinations* of $\boldsymbol{A}(\boldsymbol{r})$ and $\boldsymbol{E}(\boldsymbol{r})$ respectively.

- $\boldsymbol{A}_{\boldsymbol{k}}$ and $\boldsymbol{E}_{\boldsymbol{k}}$ are no longer Hermitian operators by themselves. Instead, their Hermitian conjugates are

$$\boldsymbol{A}_{\boldsymbol{k}}^\dagger = \boldsymbol{A}_{-\boldsymbol{k}},$$
$$\boldsymbol{E}_{\boldsymbol{k}}^\dagger = \boldsymbol{E}_{-\boldsymbol{k}}. \tag{27}$$

- Commutation relations (with $\hbar = 1$):

$$\begin{aligned}&\left[A_{i,\boldsymbol{k}},\, A^\dagger_{j,\boldsymbol{k}'}\right] = \left[-E_{i,\boldsymbol{k}},\, -E^\dagger_{j,\boldsymbol{k}'}\right] = 0,\\ &\left[A_{i,\boldsymbol{k}},\, -E^\dagger_{j,\boldsymbol{k}'}\right] = i\,\delta_{ij}\,\delta_{\boldsymbol{k}\boldsymbol{k}'}\,\mathbb{1}.\end{aligned} \tag{28}$$

Exc 6 Verify Eq. (28), given Eq. (22).

Further impose $\nabla\cdot\boldsymbol{A} = 0$ (Coulomb gauge) and $\nabla\cdot\boldsymbol{E} = 0$ (Gauss law) for electromagnetic field in the *free space*, the Fourier components $\boldsymbol{A}_{\boldsymbol{k}}$ and $\boldsymbol{E}_{\boldsymbol{k}}$ only contains the *transverse* modes, as in Eq. (18),

$$\boxed{\begin{aligned}\boldsymbol{A}_{\boldsymbol{k}} &= \sum_{\alpha=1,2} A_{\boldsymbol{k},\alpha}\,\boldsymbol{e}_{\boldsymbol{k},\alpha},\\ \boldsymbol{E}_{\boldsymbol{k}} &= \sum_{\alpha=1,2} E_{\boldsymbol{k},\alpha}\,\boldsymbol{e}_{\boldsymbol{k},\alpha},\end{aligned}} \tag{29}$$

where $\boldsymbol{e}_{\boldsymbol{k},\alpha}$ ($\alpha = 1, 2$) are orthogonal unit vectors, characterizing independent transverse polarization directions (i.e. $\boldsymbol{k}\cdot\boldsymbol{e}_{\boldsymbol{k},\alpha} = 0$).

- $A_{\boldsymbol{k},\alpha}$ and $E_{\boldsymbol{k},\alpha}$ are not Hermitian operators. As inherited from Eq. (27), their Hermitian conjugates are

$$\begin{aligned}A^\dagger_{\boldsymbol{k},\alpha} &= A_{-\boldsymbol{k},\alpha},\\ E^\dagger_{\boldsymbol{k},\alpha} &= E_{-\boldsymbol{k},\alpha}.\end{aligned} \tag{30}$$

- Commutation relations (with $\hbar = 1$):

$$\begin{aligned}&\left[A_{\boldsymbol{k},\alpha},\, A^\dagger_{\boldsymbol{k}',\alpha'}\right] = \left[-E_{\boldsymbol{k},\alpha},\, -E^\dagger_{\boldsymbol{k}',\alpha'}\right] = 0,\\ &\left[A_{\boldsymbol{k},\alpha},\, -E^\dagger_{\boldsymbol{k}',\alpha'}\right] = i\,\delta_{\alpha\alpha'}\,\delta_{\boldsymbol{k}\boldsymbol{k}'}\,\mathbb{1}.\end{aligned} \tag{31}$$

▪ Hamiltonian Operator

The Hamiltonian operator (under the Coulomb gauge)

$$\boxed{H = \frac{1}{2}\sum_{\boldsymbol{k},\alpha}\left(E^\dagger_{\boldsymbol{k},\alpha}\,E_{\boldsymbol{k},\alpha} + \omega_{\boldsymbol{k}}^2\,A^\dagger_{\boldsymbol{k},\alpha}\,A_{\boldsymbol{k},\alpha}\right),} \tag{32}$$

where $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$ is set by the dispersion relation.

Exc 7 Derive Eq. (32).

- H accounts for contributions from all possible wave modes, labeled by the **wave vector $\boldsymbol{k}$** and the **polarization index** α.
- $A_{\boldsymbol{k},\alpha}$ and $E_{\boldsymbol{k},\alpha}$ are *quantum operators* satisfying the commutation relation in Eq. (31).
  - $E^\dagger_{\boldsymbol{k},\alpha}\,E_{\boldsymbol{k},\alpha}$ originated from the $\boldsymbol{E}^2$ term, representing the *kinetic energy* of the electromagnetic field.

- $\omega_{\boldsymbol{k}}^2 A_{\boldsymbol{k},\alpha}^\dagger A_{\boldsymbol{k},\alpha}$ originated from the $\boldsymbol{B}^2$ term, representing the *potential energy* of the electromagnetic field.

Photons

For each mode of $\boldsymbol{k}$ and α, define the **photon** *creation* $a_{\boldsymbol{k},\alpha}^\dagger$ and *annihilation* $a_{\boldsymbol{k},\alpha}$ operators as

$$\begin{aligned} a_{\boldsymbol{k},\alpha} &= \frac{1}{\sqrt{2}} \left(\omega_{\boldsymbol{k}}^{1/2} A_{\boldsymbol{k},\alpha} - i\, \omega_{\boldsymbol{k}}^{-1/2} E_{\boldsymbol{k},\alpha}\right), \\ a_{\boldsymbol{k},\alpha}^\dagger &= \frac{1}{\sqrt{2}} \left(\omega_{\boldsymbol{k}}^{1/2} A_{\boldsymbol{k},\alpha}^\dagger + i\, \omega_{\boldsymbol{k}}^{-1/2} E_{\boldsymbol{k},\alpha}^\dagger\right). \end{aligned} \tag{33}$$

The inverse combination is

$$\begin{aligned} A_{\boldsymbol{k},\alpha} &= \frac{1}{\sqrt{2}}\, \omega_{\boldsymbol{k}}^{-1/2}\left(a_{\boldsymbol{k},\alpha} + a_{-\boldsymbol{k},\alpha}^\dagger\right), \\ E_{\boldsymbol{k},\alpha} &= \frac{i}{\sqrt{2}}\, \omega_{\boldsymbol{k}}^{1/2}\left(a_{\boldsymbol{k},\alpha} - a_{-\boldsymbol{k},\alpha}^\dagger\right). \end{aligned} \tag{34}$$

Exc 8 Derive Eq. (34) by inverting Eq. (33).

- They satisfy the following commutation relations

$$\begin{aligned} &[a_{\boldsymbol{k},\alpha},\, a_{\boldsymbol{k}',\alpha'}] = \left[a_{\boldsymbol{k},\alpha}^\dagger,\, a_{\boldsymbol{k}',\alpha'}^\dagger\right] = 0, \\ &\left[a_{\boldsymbol{k},\alpha},\, a_{\boldsymbol{k}',\alpha'}^\dagger\right] = \delta_{\alpha\alpha'}\, \delta_{\boldsymbol{k}\boldsymbol{k}'}\, \mathbb{1}. \end{aligned} \tag{35}$$

Exc 9 Derive Eq. (35) from Eq. (31), given the definition Eq. (33).

- **Photon number operator**:

$$n_{\boldsymbol{k},\alpha} = a_{\boldsymbol{k},\alpha}^\dagger\, a_{\boldsymbol{k},\alpha}. \tag{37}$$

  By quantum bootstrap, Eq. (35) requires that the eigenvalues of $n_{\boldsymbol{k},\alpha}$ are quantized to natural numbers

$$n_{\boldsymbol{k},\alpha} = 0, 1, 2, \ldots \in \mathbb{N}. \tag{38}$$

- **Photons are Bosons**: their creation $a_{\boldsymbol{k},\alpha}^\dagger$ and annihilation $a_{\boldsymbol{k},\alpha}$ operators satisfy the *bosonic* commutation relations, and each photon mode (labeled by $\boldsymbol{k}$ and α) can be occupied by an arbitrary number $n_{\boldsymbol{k},\alpha}$ of photons.

Photon Energy

The total **energy** of the electromagnetic field is

$$H = \int d^3 \boldsymbol{r} \, \frac{1}{2} \left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right). \tag{39}$$

From Eq. (32), H can be written as

$$H = \sum_{\boldsymbol{k},\alpha} \omega_{\boldsymbol{k}} \left(n_{\boldsymbol{k},\alpha} + \frac{1}{2}\right). \tag{40}$$

- Every photon mode is (mathematically) equivalent to a simple harmonic oscillator with *quantized* energy levels.
- Adding/removing each photon ($n_{\boldsymbol{k},\alpha} \to n_{\boldsymbol{k},\alpha} \pm 1$) will cause the total energy H to increase/decrease by $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$ (or $\hbar\, \omega_{\boldsymbol{k}} = \hbar\, c\, |\boldsymbol{k}|$, if the units are restored).

  ⇒ **Energy quantization**: each photon of *wave vector* $\boldsymbol{k}$ and *polarization* α carries $\hbar\, \omega_{\boldsymbol{k}}$ unit of *energy*.
- Even if $n_{\boldsymbol{k},\alpha} = 0$ (in the photon **vacuum state**), there is still $\frac{1}{2}\, \hbar\, \omega_{\boldsymbol{k}}$ energy associated with each photon mode, known as the **vacuum energy**.

$$E_{\text{vac}} = \sum_{\boldsymbol{k},\alpha} \frac{\omega_{\boldsymbol{k}}}{2} = \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}}. \tag{41}$$

▪ Field Operators

The field operators can be recovered in terms of photon operators,

- Vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{\sqrt{2}} \sum_{\boldsymbol{k},\alpha} \omega_{\boldsymbol{k}}^{-1/2}\, \boldsymbol{e}_{\boldsymbol{k},\alpha} \left(a_{\boldsymbol{k},\alpha}\, e^{i \boldsymbol{k}\cdot\boldsymbol{r}} + a_{\boldsymbol{k},\alpha}^{\dagger}\, e^{-i \boldsymbol{k}\cdot\boldsymbol{r}}\right). \tag{42}$$

- Electromagnetic field

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= \frac{i}{\sqrt{2}} \sum_{\boldsymbol{k},\alpha} \omega_{\boldsymbol{k}}^{1/2}\, \boldsymbol{e}_{\boldsymbol{k},\alpha} \left(a_{\boldsymbol{k},\alpha}\, e^{i \boldsymbol{k}\cdot\boldsymbol{r}} - a_{\boldsymbol{k},\alpha}^{\dagger}\, e^{-i \boldsymbol{k}\cdot\boldsymbol{r}}\right), \\ \boldsymbol{B}(\boldsymbol{r}) &= \frac{i}{\sqrt{2}} \sum_{\boldsymbol{k},\alpha} \omega_{\boldsymbol{k}}^{-1/2}\, \boldsymbol{k} \times \boldsymbol{e}_{\boldsymbol{k},\alpha} \left(a_{\boldsymbol{k},\alpha}\, e^{i \boldsymbol{k}\cdot\boldsymbol{r}} - a_{\boldsymbol{k},\alpha}^{\dagger}\, e^{-i \boldsymbol{k}\cdot\boldsymbol{r}}\right). \end{aligned} \tag{43}$$

Exc 10 | Verify Eq. (42) and Eq. (43).

▪ Photon Momentum

The total **momentum** carried by the electromagnetic field (a.k.a. the Poynting vector) is

$$\boldsymbol{K} = \int d^3\boldsymbol{r}\, \boldsymbol{E}\times\boldsymbol{B}. \tag{44}$$

Substitute Eq. (43) into Eq. (44),

$$\boxed{\boldsymbol{K} = \sum_{\boldsymbol{k},\alpha} \boldsymbol{k}\, n_{\boldsymbol{k},\alpha}.} \tag{45}$$

Exc 11 Derive Eq. (45).

- Adding/removing each photon ($n_{\boldsymbol{k},\alpha} \to n_{\boldsymbol{k},\alpha} \pm 1$) will cause the total momentum $\boldsymbol{K}$ to increase/decrease by $\boldsymbol{k}$ (or $\hbar\,\boldsymbol{k}$, if the units are restored).

  ⇒ Each photon of *wave vector* $\boldsymbol{k}$ and *polarization* α carries $\hbar\,\boldsymbol{k}$ amount of *momentum*.

▪ Photon Spin

The total **spin angular momentum** carried by the electromagnetic field is

$$\boldsymbol{S} = \int d^3\boldsymbol{r}\, \boldsymbol{E}\times\boldsymbol{A}. \tag{46}$$

Substitute Eq. (43) into Eq. (46),

$$\boxed{\boldsymbol{S} = \sum_{\boldsymbol{k},\alpha} \frac{i\,\boldsymbol{k}}{|\boldsymbol{k}|} \left(a_{\boldsymbol{k},1}\, a^\dagger_{\boldsymbol{k},2} - a^\dagger_{\boldsymbol{k},1}\, a_{\boldsymbol{k},2}\right).} \tag{47}$$

Exc 12 Derive Eq. (47).

The spin operator $\boldsymbol{S}$ is not diagonal in the photon polarization space, i.e. it mixes different polarization modes.

- Define the **circular polarization** basis

$$\begin{aligned} a_{\boldsymbol{k},\pm} &= \frac{1}{\sqrt{2}} \left(a_{\boldsymbol{k},1} + i\, a_{\boldsymbol{k},2}\right), \\ a^\dagger_{\boldsymbol{k},\pm} &= \frac{1}{\sqrt{2}} \left(a^\dagger_{\boldsymbol{k},1} - i\, a^\dagger_{\boldsymbol{k},2}\right), \end{aligned} \tag{48}$$

  where ± labels the left/right circular polarized light.

The spin operator is now diagonalized

$$\boxed{\boldsymbol{S} = \sum_{\boldsymbol{k},\alpha} \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \left(n_{\boldsymbol{k},+} - n_{\boldsymbol{k},-}\right),} \tag{49}$$

where $n_{\boldsymbol{k},\pm} = a^\dagger_{\boldsymbol{k},\pm}\, a_{\boldsymbol{k},\pm}$ is the number of left/right circular polarized photons.

- Each left/right circular polarized photon carries **spin-1 angular momentum** along/against the *wave vector* direction $\boldsymbol{k}/|\boldsymbol{k}|$ (the light propagation direction).

Quantum Vacuum Fluctuations

Uncertainty Relation

There is an **uncertainty relation** between the electric and magnetic fields

$$\frac{\operatorname{var} E_{\boldsymbol{k}} + \operatorname{var} E_{-\boldsymbol{k}}}{2} \frac{\operatorname{var} B_{\boldsymbol{k}'} + \operatorname{var} B_{-\boldsymbol{k}'}}{2} \geq \omega_{\boldsymbol{k}}^2 \delta_{\boldsymbol{k}\boldsymbol{k}'}, \tag{50}$$

where the **variance** can be defined for the *electric* ($\operatorname{var} E_{\boldsymbol{k}}$) and the *magnetic* ($\operatorname{var} B_{\boldsymbol{k}}$) field respectively at any wave vector $\boldsymbol{k}$,

$$\begin{aligned} \operatorname{var} E_{\boldsymbol{k}} &:= \langle \boldsymbol{E}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{E}_{\boldsymbol{k}} \rangle - \langle \boldsymbol{E}_{\boldsymbol{k}}^\dagger \rangle \cdot \langle \boldsymbol{E}_{\boldsymbol{k}} \rangle, \\ \operatorname{var} B_{\boldsymbol{k}} &:= \langle \boldsymbol{B}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{B}_{\boldsymbol{k}} \rangle - \langle \boldsymbol{B}_{\boldsymbol{k}}^\dagger \rangle \cdot \langle \boldsymbol{B}_{\boldsymbol{k}} \rangle. \end{aligned} \tag{51}$$

Exc 13 Prove the uncertainty relation Eq. (50).

- **Noise trade-off**: It highlights the *inherent quantum noise* present in electromagnetic fields — *reducing* the noise in one field ($\boldsymbol{E}$ or $\boldsymbol{B}$) inevitably leads to an *increase* in the noise of the other.
- **Frequency dependence**: The bound $\omega_{\boldsymbol{k}}^2$ grows with the mode frequency — *higher frequency* field exhibits *stronger quantum noise*.

Let us check the uncertainty relation explicitly on the **photon number eigenstate** (Fock state)

$$|n_{\boldsymbol{k},1}, n_{\boldsymbol{k},2}; n_{-\boldsymbol{k},1}, n_{-\boldsymbol{k},2}\rangle. \tag{56}$$

- The uncertainties of electric and magnetic fields are given by

$$\begin{aligned} \operatorname{var} E_{\boldsymbol{k}} &= \\ \operatorname{var} B_{\boldsymbol{k}} &= \sum_\alpha \frac{\omega_{\boldsymbol{k}}}{2} (n_{\boldsymbol{k},\alpha} + n_{-\boldsymbol{k},\alpha} + 1), \end{aligned} \tag{57}$$

therefore the uncertainty relation holds for all Fock states ($\forall\, n_{\boldsymbol{k},\alpha} \in \mathbb{N}$)

$$\operatorname{var} E_{\boldsymbol{k}} \operatorname{var} B_{\boldsymbol{k}} = \omega_{\boldsymbol{k}}^2 \left(\sum_\alpha \frac{1}{2} (n_{\boldsymbol{k},\alpha} + n_{-\boldsymbol{k},\alpha} + 1) \right)^2 \geq \omega_{\boldsymbol{k}}^2. \tag{58}$$

Exc 14 Calculate the variances in Eq. (57) on the state Eq. (56).

- Specifically, the uncertainty relation is *saturated* when

$$\forall\ : n_{\boldsymbol{k},\alpha} = 0, \tag{59}$$

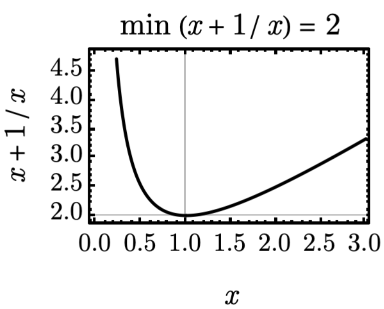
i.e. on the photon **vacuum state** $|\text{vac}\rangle = |0,0;0,0\rangle$. We say $|\text{vac}\rangle$ is a **minimal uncertainty state**.

- The finite amount of **vacuum energy** is also a consequence of the uncertainty relation. Given that

$$\left\langle \boldsymbol{E}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{E}_{\boldsymbol{k}} \right\rangle \left\langle \boldsymbol{B}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{B}_{\boldsymbol{k}} \right\rangle \geq \operatorname{var} E_{\boldsymbol{k}} \operatorname{var} B_{\boldsymbol{k}} \geq \omega_{\boldsymbol{k}}^2, \tag{60}$$

The total energy is therefore bounded

$$\begin{aligned} E = \langle H \rangle &= \frac{1}{2} \sum_{\boldsymbol{k}} \left(\left\langle \boldsymbol{E}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{E}_{\boldsymbol{k}} \right\rangle + \left\langle \boldsymbol{B}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{B}_{\boldsymbol{k}} \right\rangle \right) \\ &\geq \frac{1}{2} \sum_{\boldsymbol{k}} \left(\left\langle \boldsymbol{E}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{E}_{\boldsymbol{k}} \right\rangle + \frac{\omega_{\boldsymbol{k}}^2}{\left\langle \boldsymbol{E}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{E}_{\boldsymbol{k}} \right\rangle} \right) \\ &\overset{x_{\boldsymbol{k}} = \left\langle \boldsymbol{E}_{\boldsymbol{k}}^\dagger \cdot \boldsymbol{E}_{\boldsymbol{k}} \right\rangle / \omega_{\boldsymbol{k}}}{=} \sum_{\boldsymbol{k}} \frac{\omega_{\boldsymbol{k}}}{2} \left(x_{\boldsymbol{k}} + \frac{1}{x_{\boldsymbol{k}}} \right) \\ &\geq \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = E_{\text{vac}}. \end{aligned} \tag{61}$$



The lower bound turns out to match the vacuum energy E_{vac}, as discussed in Eq. (41).

▪ Casimir Effect: Vacuum Energy is Real

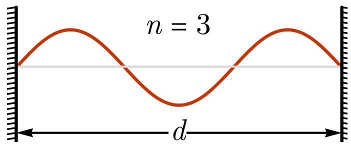
The $\omega_{\boldsymbol{k}}/2$ **vacuum energy** associated with each photon mode is real and has a *measurable* physical effect—the **Casimir effect**: two *uncharged*, parallel conducting plates in vacuum experience an *attractive* force due to *quantum vacuum fluctuations* of the electromagnetic field.

To avoid the complications, we are going to demonstrate the effect

- in $(1+1)$ *spacetime dimension*,
- and for *scalar* field (like for sound waves).

Two plates (walls) separated by distance d. The standing wave between the plates:

$$\psi_n(x) = \sin\left(\frac{n\,\pi\,x}{d}\right). \tag{62}$$

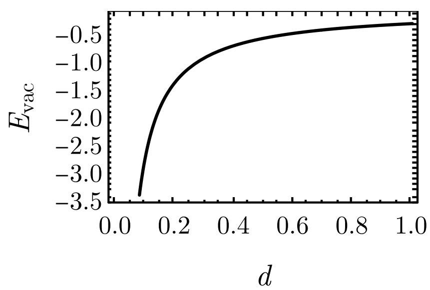


- $n = 1, 2, 3, \ldots$ is the **mode index**, labeling different wave modes on which bosons (like phonons) can occupy.
- Oscillation frequency (assuming linear dispersion)

$$\omega_n = k_n = \frac{n\,\pi}{d}. \tag{63}$$

The vacuum energy between the two plates:

$$\boxed{E_{\text{vac}}(d) = \sum_{n=1}^{\infty} \omega_n = \frac{\pi}{d} \sum_{n=1}^{\infty} n \overset{!?}{=} -\frac{\pi}{12\,d}.} \tag{64}$$



The energy E_{vac} is lower when the plates are closer (smaller d) — the *quantum vacuum fluctuations* wants to pull the plates together, mediating an *attractive* force between the plates!

How on earth can the sum of all natural numbers be $-\frac{1}{12}$?

- A brute-force yet incorrect answer (by Srinivasa Ramanujan):

$$\begin{array}{rcl} S & = & 1 + 2 + 3 + 4 + 5 + 6 + \ldots \\ 4\,S & = & \quad\;\; 4 \quad\;\; + 8 \quad\;\; + 12 + \ldots \\ -3\,S & = & 1 - 2 + 3 - 4 + 5 - 6 + \ldots \\ & \overset{x=1}{=} & 1 - 2\,x + 3\,x^2 - 4\,x^3 + 5\,x^4 - 6\,x^5 + \ldots \\ & = & (1+x)^{-2} \\ & \overset{x=1}{=} & 1/4 \end{array} \tag{65}$$

  Therefore, $S = -1/12$. However, S is *not* a *convergent* series to start with, so the above manipulations are illegitimate.

- **Regularization** — a correct understanding (by Terence Tao, see Ref. [1]): introduce a *regularization function* $\eta(n/N)$ that smoothly suppresses large terms in the summation across a *cut-off scale* N, while approximates the original divergent series as $N \to \infty$:

$$\sum_{n=1}^{\infty} n\,\eta(n/N) \xrightarrow{N\to\infty} \sum_{n=1}^{\infty} n. \tag{66}$$

Examples:

$$\begin{aligned}
\sum_{n=1}^{\infty} n\,e^{-n/N} &= N^2 - \frac{1}{12} + \frac{1}{240\,N^2} + O\left(\frac{1}{N^4}\right),\\
\sum_{n=1}^{\infty} \frac{n}{(n/N)^3+1} &= \frac{2\pi N^2}{3\sqrt{3}} - \frac{1}{12} + O\left(\frac{1}{N^4}\right),\\
\sum_{n=1}^{\infty} \frac{n}{(n/N)^4+1} &= \frac{\pi N^2}{4} - \frac{1}{12} + O\left(\frac{1}{N^4}\right),\\
\sum_{n=1}^{\infty} \frac{n}{\left((n/N)^2+1\right)^2} &= \frac{N^2}{2} - \frac{1}{12} - \frac{1}{60\,N^2} + O\left(\frac{1}{N^4}\right),\\
&\ldots
\end{aligned} \tag{67}$$

Exc 15 Verify Eq. (67) by *Mathematica*.

- The *leading* term diverge as N^2 (confirming the divergent nature of the series), but its coefficient *depends* on the choice of the regularization function $\eta(n/N)$, making it **non-universal**.
- The *sub-leading* term is always $-1/12$, which is **universal**, *independent* of the choice of regularization. — This represents an *intrinsic and invariant feature* hidden beneath the apparent divergence of the series.

In the context of Casimir effect, the *non-universal regularization* reflects our *ignorance* about the behavior of *high-frequency modes* in a physical system.

- In reality, the mode frequency can grow indefinitely towards infinity (e.g. materials impose a natural cutoff at plasma frequency), the frequency summation must be regularized

$$E_{\text{vac}}(d) = \sum_{n=1}^{\infty} \omega_n\, e^{-a\,\omega_n} = \frac{d}{\pi\,a^2} - \frac{\pi}{12\,d} + \ldots. \tag{68}$$

- Consider the vacuum energy both *inside* and *outside* the plates,

$$E_{\text{vac}}^{\text{total}}(d) = E_{\text{vac}}(d) + E_{\text{vac}}(L-d), \tag{69}$$

assuming $L \to \infty$ is the size of the full space outside, the **attractive force** between the plates is the only thing we can measure

$$\begin{aligned}
F_{\text{Casimir}}(d) &= -\partial_d\, E_{\text{vac}}^{\text{total}}(d)\\
&= \left(-\frac{1}{\pi\,a^2} - \frac{\pi}{24\,d^2} + \ldots\right) + \left(\frac{1}{\pi\,a^2} + \frac{\pi}{24\,(L-d)^2} + \ldots\right)
\end{aligned} \tag{70}$$

$$\overset{a\to 0, L\to\infty}{=} -\frac{\pi}{24\, d^2},$$

where the leading non-universal divergence cancels exactly.

The Casimir force between conducting plates has been *experimentally* measured and confirmed [2,3]. — $1 + 2 + 3 + \ldots = -1/12$ is real!

[1] Terence Tao. The Euler-Maclaurin formula, Bernoulli numbers, the zeta function, and real-variable analytic continuation. (2010)

[2] S. K. Lamoreaux. Demonstration of the Casimir Force in the 0.6 to 6 μm Range. Phys. Rev. Lett. 78, 5 (1998).

[3] Umar Mohideen, Anushree Roy. Precision Measurement of the Casimir Force from 0.1 to 0.9 microns. arXiv:physics/9805038 (1998).

Quantum Coherence of Light

Light-Matter Interaction

Dipole Approximation

In most situations, the **light-matter coupling** is through the *electric field* $\boldsymbol{E}$ interacting with a *dipole moment* $\boldsymbol{d}$ of atoms or molecules, described by the Hamiltonian

$$\boxed{H_{\text{cp}} = -\boldsymbol{E}\cdot\boldsymbol{d}.} \tag{71}$$

The electric field operator is given by Eq. (43)

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{i}{\sqrt{2}} \sum_{\boldsymbol{k},\alpha} \omega_{\boldsymbol{k}}^{1/2}\, \boldsymbol{e}_{\boldsymbol{k},\alpha}\left(a_{\boldsymbol{k},\alpha}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - a_{\boldsymbol{k},\alpha}^{\dagger}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right). \tag{72}$$

- **Mode Parity**: Eq. (72) can be further decomposed into

$$\boldsymbol{E}\,(\boldsymbol{r}) = \boldsymbol{E}_{\text{even}}(\boldsymbol{r}) + \boldsymbol{E}_{\text{odd}}(\boldsymbol{r}), \tag{73}$$

  - the *even* parity part (i.e. $\boldsymbol{E}_{\text{even}}(\boldsymbol{r}) = \boldsymbol{E}_{\text{even}}(-\boldsymbol{r})$)

$$\boldsymbol{E}_{\text{even}}(\boldsymbol{r}) = \frac{\omega_{\boldsymbol{k}}^{1/2}}{\sqrt{2}} \cos(\boldsymbol{k}\cdot\boldsymbol{r})\, \boldsymbol{e}_{\boldsymbol{k},\alpha}\, i\left(a_{\boldsymbol{k},\alpha} - a_{\boldsymbol{k},\alpha}^{\dagger}\right), \tag{74}$$

  - the *odd* parity part (i.e. $\boldsymbol{E}_{\text{odd}}(\boldsymbol{r}) = -\boldsymbol{E}_{\text{odd}}(-\boldsymbol{r})$)

$$\boldsymbol{E}_{\text{odd}}(\boldsymbol{r}) = -\frac{\omega_{\boldsymbol{k}}^{1/2}}{\sqrt{2}} \sin(\boldsymbol{k}\cdot\boldsymbol{r})\, \boldsymbol{e}_{\boldsymbol{k},\alpha}\left(a_{\boldsymbol{k},\alpha} + a_{\boldsymbol{k},\alpha}^{\dagger}\right). \tag{75}$$

- **Single-Mode Approximation**: focus on a specific mode of fixed wavelength, parity (e.g. odd), and polarization,

$$\boldsymbol{E} = -\frac{\omega^{1/2}}{\sqrt{2}} \sin(\boldsymbol{k}\cdot\boldsymbol{r})\, \boldsymbol{e}\left(a + a^\dagger\right). \tag{76}$$

- ω - **photon energy** (mode frequency),
- $\boldsymbol{e}$ - the **polarization vector** of the mode,
- $a^\dagger$, a - the **creation** and **annihilation operators** of the mode, such that the electromagnetic field energy is given by

$$H_{\text{em}} = \omega\left(a^\dagger\, a + \frac{1}{2}\right) \tag{77}$$

Under Eq. (76), the light-matter coupling Hamiltonian in Eq. (71) becomes

$$H_{\text{cp}} = g\, d\left(a + a^\dagger\right), \tag{78}$$

where

- g - the light-matter **coupling strength** (can have spacial dependence, following the mode profile in the space)

$$g = \left(\frac{\omega}{2}\right)^{1/2} \sin(\boldsymbol{k}\cdot\boldsymbol{r}) \tag{79}$$

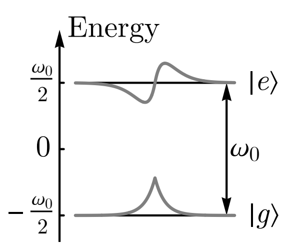
- d - **dipole operator** describing how matter responses to electric field along the polarization $\boldsymbol{e}$ direction:

$$d = \boldsymbol{e}\cdot\boldsymbol{d}. \tag{80}$$

A Two-Level Atom

Atoms are building blocks of matter. The electromagnetic field largely *electrons* within them. **Electrons** occupy discrete *energy levels*, and transitions between these levels are responsible for *emission* and *absorption* of light.

In many situations, it is sufficient to consider only two relevant levels of an atom:



- $|g\rangle$ - **ground state** of the atom,
- $|e\rangle$ - **exited state** of the atom,

- $\omega_0 = E_e - E_g$ - atom **excitation energy** (energy level splitting, transition frequency).

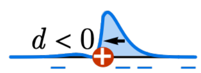
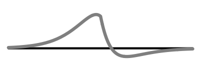
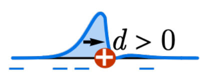
This simplification leads to the concept of a **two-level atom** — also viewed as a *qubit*, for which *Pauli operators* can be defined:

$$\boxed{\begin{aligned} \sigma^z &= |e\rangle\langle e| - |g\rangle\langle g|, \\ \sigma^x &= |g\rangle\langle e| + |e\rangle\langle g|. \end{aligned}} \tag{81}$$

- **Inversion operator**: σ^z splits the energy levels between $|g\rangle$ and $|e\rangle$,

$$\boxed{H_{\text{atm}} = \frac{\omega_0}{2}\,\sigma^z.} \tag{82}$$

- **Transition operator**: σ^x mixes $|g\rangle$ and $|e\rangle$, leading to different *dipole moments* (by deforming the electron density):

Eigenvalue	Eigenstate	Wave func.	Density dist.
$\sigma^x = +1$	$\frac{1}{\sqrt{2}}(\lvert g\rangle + \lvert e\rangle)$		
$\sigma^x = -1$	$\frac{1}{\sqrt{2}}(\lvert g\rangle - \lvert e\rangle)$		

This implies $d \propto -\sigma^x$ (any proportionality constant here can be absorbed into the coupling constant g below), and the light-matter compiling Hamiltonian in Eq. (78) becomes

$$\boxed{H_{\text{cp}} = -g\,\sigma^x\left(a^\dagger + a\right),} \tag{83}$$

for the two-level atom.

▪ Jaynes-Cummings Model

The Jaynes-Cummings model is a simplified model, describing the interaction between a **two-level atom** (i.e. a qubit) and a *single mode* **electromagnetic field**.

Put together Eq. (77), Eq. (82) and Eq. (83), the Jaynes-Cummings model is given by the total Hamiltonian

$$\boxed{H = H_{\text{atm}} + H_{\text{em}} + H_{\text{cp}},} \tag{84}$$

which reads

$$H = \frac{\omega_0}{2}\,\sigma^z + \omega\left(a^\dagger\, a + \frac{1}{2}\right) - g\,\sigma^x\left(a^\dagger + a\right). \tag{85}$$

- **Raising and Lowering Operators**: Introduce $\sigma^\pm$ to raise and lower the electron between levels

$$\boxed{\begin{aligned}\sigma^{+} &= |e\rangle\langle g|,\\ \sigma^{-} &= |g\rangle\langle e|,\end{aligned}} \tag{86}$$

such that, by definition Eq. (81),

$$\sigma^{x} = \sigma^{+} + \sigma^{-}. \tag{87}$$

- **Decoupled Operator Dynamics**: In the decoupled limit $g \to 0$, the operators evolves as

$$\begin{aligned} &a(t) = a(0)\, e^{-i\,\omega\, t},\; a^{\dagger}(t) = a^{\dagger}(0)\, e^{i\,\omega\, t};\\ &\sigma^{\pm}(t) = \sigma^{\pm}(0)\, e^{\pm i\,\omega_0\, t}. \end{aligned} \tag{88}$$

Exc 16 Show that Eq. (88) satisfies the Heisenberg equation $\partial_t A = i\,[H, A]$ that governs the operator dynamics, for H in the $g \to 0$ limit.

- **Rotating Wave Approximation**: The light-matter coupling Hamiltonian H_{cp} can be expanded as

$$\begin{aligned} H_{\text{cp}} &= -g\,(\sigma^{+} + \sigma^{-})\left(a^{\dagger} + a\right)\\ &= -g\left(\sigma^{+}\, a^{\dagger} + \sigma^{-}\, a^{\dagger} + \sigma^{+}\, a + \sigma^{-}\, a\right). \end{aligned} \tag{89}$$

For small g, the approximate time dependence of these terms are

$$\begin{aligned} &\sigma^{+}\, a^{\dagger} \sim e^{i\,(+\omega_0+\omega)\,t},\\ &\sigma^{-}\, a^{\dagger} \sim e^{i\,(-\omega_0+\omega)\,t},\\ &\sigma^{+}\, a \sim e^{i\,(+\omega_0-\omega)\,t},\\ &\sigma^{-}\, a \sim e^{i\,(-\omega_0-\omega)\,t}. \end{aligned} \tag{90}$$

When the **level spacing** ω_0 and the **photon frequency** ω are comparable (i.e. $\omega_0 \approx \omega$), $e^{\pm i\,(\omega_0+\omega)\,t}$ ($\sigma^{+}\, a^{\dagger}$ and $\sigma^{-}\, a$) oscillate much more rapidly than $e^{\pm i(\omega_0-\omega)\,t}$ ($\sigma^{-}\, a^{\dagger}$ and $\sigma^{+}\, a$), leading to their effect averaging out to zero over time. Thus H_{cp} reduces to

$$H_{\text{cp}} = -g\left(\sigma^{-}\, a^{\dagger} + \sigma^{+}\, a\right), \tag{91}$$

under the rotating wave approximation.

- $\sigma^{-}\, a^{\dagger} := |g\rangle\langle e| \otimes a^{\dagger}$ - atom decays from $|e\rangle$ to $|g\rangle$ to emit a photon.
- $\sigma^{+}\, a := |e\rangle\langle g| \otimes a$ - atom excites from $|g\rangle$ to $|e\rangle$ to absorb a photon.

The total Hamiltonian Eq. (85) reduces to

$$\boxed{H = \frac{\omega_0}{2}\,\sigma^{z} + \omega\left(a^{\dagger}\, a + \frac{1}{2}\right) - g\left(\sigma^{-}\, a^{\dagger} + \sigma^{+}\, a\right),} \tag{92}$$

which is widely referred to as the **Jaynes-Cummings model**.

▪ Rabi Oscillation

A two-level atom can *emit* and *absorb* a photon, exchanging energy *coherently* with the electromagnetic field in an *oscillatory* manner, leading to *periodic transitions* between its *ground* and *excited* states. — a phenomenon known as the **Rabi oscillation**.

To understand Rabi oscillation, consider two relevant states

- $|e\rangle \otimes |n\rangle$: excited state atom with n photons,
- $|g\rangle \otimes |n+1\rangle$: ground state atom with $n+1$ photons.

They span a 2-dimensional Hilbert space,

$$\mathcal{H} = \mathrm{span}\{|e\rangle \otimes |n\rangle,\ |g\rangle \otimes |n+1\rangle\}. \tag{93}$$

in which the **Jaynes-Cummings model** can be represented as

$$H \simeq \begin{pmatrix} \frac{\omega_0}{2} + \omega\left(n+\frac{1}{2}\right) & -g\sqrt{n+1} \\ -g\sqrt{n+1} & -\frac{\omega_0}{2} + \omega\left(n+\frac{3}{2}\right) \end{pmatrix}, \tag{94}$$

or expanded in terms of Pauli matrices

$$\boxed{H = E_0\, I + \frac{\Delta}{2}\, Z - g\sqrt{n+1}\; X,} \tag{95}$$

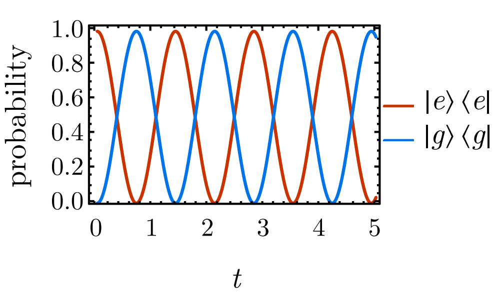
with

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{96}$$

where

- $E_0 := \omega\,(n+1)$ is a background energy of the system,
- $\Delta := \omega_0 - \omega$ is difference between the atomic *level resonant frequency* ω_0 and the *photon frequency* ω, also called the **detuning**.
- g is the *coupling strength*, and n is the photon number.

Starting from the initial state $|e\rangle \otimes |n\rangle$:



- The **Rabi oscillation frequency** is given by

$$\Omega = \sqrt{\Delta^2 + 4\, g^2 (n+1)}\,. \tag{97}$$

Exc 17 Prove Eq. (97).

- **Vacuum Rabi oscillations**: Ω remains finite even if $n = 0 \Rightarrow$ Rabi oscillation can occur in vacuum (typically inside a *high-quality optical cavity*).

Coherent State

Single-Mode Photon

Let us focus on a single photon mode. Eq. (42) and Eq. (43) are reduced to

$$\begin{aligned} \boldsymbol{A} &= \frac{1}{\sqrt{2}}\, \omega^{-1/2}\, \boldsymbol{e}\left(a\, e^{i \boldsymbol{k}\cdot\boldsymbol{r}} + a^\dagger\, e^{-i \boldsymbol{k}\cdot\boldsymbol{r}}\right), \\ \boldsymbol{E} &= \frac{i}{\sqrt{2}}\, \omega^{1/2}\, \boldsymbol{e}\left(a\, e^{i \boldsymbol{k}\cdot\boldsymbol{r}} - a^\dagger\, e^{-i \boldsymbol{k}\cdot\boldsymbol{r}}\right), \\ \boldsymbol{B} &= \frac{i}{\sqrt{2}}\, \omega^{-1/2}\, \boldsymbol{k}\times\boldsymbol{e}\left(a\, e^{i \boldsymbol{k}\cdot\boldsymbol{r}} - a^\dagger\, e^{-i \boldsymbol{k}\cdot\boldsymbol{r}}\right). \end{aligned} \tag{98}$$

- a, $a^\dagger$ - photon annihilation, creation operators, satisfying

$$\left[a, a^\dagger\right] = \mathbb{1}. \tag{99}$$

The photon vacuum state is defined by

$$a\,|\text{vac}\rangle = 0. \tag{100}$$

We already known that the vacuum state $|\text{vac}\rangle$ is a minimal uncertainty state of the electromagnetic field.

Definition

Are there any other minimal uncertainty states besides $|\text{vac}\rangle$?

Yes, they are known as the **coherent state** (or called Glauber state). Each coherent state $|\alpha\rangle$ is labeled by a *complex number* $\alpha \in \mathbb{C}$ and defined as the the *eigenstate* of the *annihilation operator* a with the *eigenvalue* α.

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle. \tag{101}$$

Note that the operator a is *non-Hermitian*,

- its eigenvalues $\alpha \in \mathbb{C}$ can be *complex*,
- its eigenstates with different eigenvalues may *not* be *orthogonal*, i.e. $\langle \alpha_1 | \alpha_2 \rangle \neq \delta(\alpha_1 - \alpha_2)$.

- Nevertheless, we do assume that $|\alpha\rangle$ is *normalized*, i.e. $\langle\alpha|\alpha\rangle = 1$.

Eq. (101) also implies

$$\langle\alpha|\, a^\dagger = \langle\alpha|\, \alpha^*. \tag{102}$$

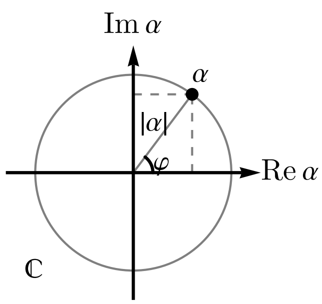
Eq. (101) and Eq. (102) enables us to evaluate operator expectation values conveniently on the coherent state:

$$\begin{aligned}
&\langle\alpha|\, a\,|\alpha\rangle = \alpha,\\
&\langle\alpha|\, a^\dagger\,|\alpha\rangle = \alpha^*,\\
&\langle\alpha|\, a^n\,|\alpha\rangle = \alpha^n,\\
&\langle\alpha|\left(a^\dagger\right)^n|\alpha\rangle = (\alpha^*)^n,\\
&\langle\alpha|\, a^\dagger\, a\,|\alpha\rangle = \alpha^*\,\alpha = |\alpha|^2,\\
&\langle\alpha|\, a\;\, a^\dagger\,|\alpha\rangle = \langle\alpha|\left(a^\dagger\, a + \mathbb{1}\right)|\alpha\rangle = |\alpha|^2 + 1.
\end{aligned} \tag{103}$$

▪ Physical Properties

Assuming the complex parameter α admits the polar decomposition

$$\alpha = |\alpha|\, e^{i\,\varphi}. \tag{104}$$



The observable expectation values on the coherent state $|\alpha\rangle$ are

- Linear properties in fields:

$$\begin{aligned}
&\langle\alpha|\, \boldsymbol{A}\,|\alpha\rangle = (2/\omega)^{1/2}\, \boldsymbol{e}\,|\alpha|\cos(\boldsymbol{k}\cdot\boldsymbol{r} + \varphi),\\
&\langle\alpha|\, \boldsymbol{E}\,|\alpha\rangle = -(2\,\omega)^{1/2}\, \boldsymbol{e}\,|\alpha|\sin(\boldsymbol{k}\cdot\boldsymbol{r} + \varphi),\\
&\langle\alpha|\, \boldsymbol{B}\,|\alpha\rangle = -(2/\omega)^{1/2}\, \boldsymbol{k}\times\boldsymbol{e}\,|\alpha|\sin(\boldsymbol{k}\cdot\boldsymbol{r} + \varphi),
\end{aligned} \tag{105}$$

Exc 18 Derive Eq. (105) using Eq. (103).

The *coherent state* $|\alpha\rangle$ of a *photon mode* (of wave vector $\boldsymbol{k}$ and polarization $\boldsymbol{e}$) describes a *snapshot* of electromagnetic wave in the *space* with

- $|\alpha|$ - **wave amplitude**,
- φ - **phase** of the wave.

- Quadratic properties in fields:

$$\langle\alpha|\, \boldsymbol{E}^\dagger \cdot \boldsymbol{E}\, |\alpha\rangle = \langle\alpha|\, \boldsymbol{B}^\dagger \cdot \boldsymbol{B}\, |\alpha\rangle = 2\,\omega\, |\alpha|^2 \sin^2(\boldsymbol{k}\cdot\boldsymbol{r}+\varphi) + \frac{\omega}{2}. \quad (106)$$

Exc 19 Derive Eq. (106) using Eq. (103).

Therefore, for single-mode photon coherent state,

$$\begin{aligned} \operatorname{var} E &= \langle\alpha|\, \boldsymbol{E}^\dagger \cdot \boldsymbol{E}\, |\alpha\rangle - \langle\alpha|\, \boldsymbol{E}^\dagger\, |\alpha\rangle \cdot \langle\alpha|\, \boldsymbol{E}\, |\alpha\rangle = \frac{\omega}{2}, \\ \operatorname{var} B &= \langle\alpha|\, \boldsymbol{B}^\dagger \cdot \boldsymbol{B}\, |\alpha\rangle - \langle\alpha|\, \boldsymbol{B}^\dagger\, |\alpha\rangle \cdot \langle\alpha|\, \boldsymbol{B}\, |\alpha\rangle = \frac{\omega}{2}. \end{aligned} \quad (107)$$

Saturating the uncertainty bound (for single-mode)

$$\operatorname{var} E \operatorname{var} B \geq (\omega/2)^2. \quad (108)$$

Note: if we consider two polarization mode for each wave vector $\boldsymbol{k}$, we would have $\operatorname{var} E_{\boldsymbol{k}} = \operatorname{var} B_{\boldsymbol{k}} = \omega_{\boldsymbol{k}}$, thereby saturating the uncertainty bound $\operatorname{var} E_{\boldsymbol{k}} \operatorname{var} B_{\boldsymbol{k}} \geq \omega_{\boldsymbol{k}}^2$in Eq. (50).

Conclusion: All **coherent states** are **minimal uncertainty states** (regardless of the parameter α) — they are the "most *classical*" *quantum* states, with minimal quantum fluctuations.

▪ Fock State Representation

In terms of the Fock state basis $|n\rangle$, a coherent state $|\alpha\rangle$ can be represented as

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle. \quad (109)$$

Exc 20 Verify Eq. (109) by showing that $|\alpha\rangle$ constructed this way satisfies the definition Eq. (101).

- In particular, the *vacuum state* $|\text{vac}\rangle := |n{=}0\rangle$ is also a coherent state with $\alpha = 0$, and admits minimal uncertainty.

HW 1

Use Eq. (109) to show:

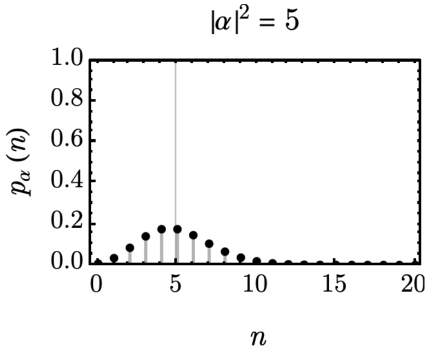
(i) the scalar product between two coherent states is given by $\langle\alpha|\beta\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha^*\beta}$,

(ii) such that the *transition probability* between states $|\alpha\rangle$ and $|\beta\rangle$ decays with the distance $|\alpha-\beta|$ in the complex plane as a Gaussian function, i.e. $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$.

Lesson: although coherent states are not strictly orthogonal, as long as their complex parameters are sufficiently separated, their inner product becomes negligible, i.e. they are approximately orthogonal.

- Based on Eq. (109), the probability to observe n photons in the coherent state $|\alpha\rangle$ is given by

$$p_\alpha(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2\,n}}{n!}\, e^{-|\alpha|^2}. \tag{110}$$



- The **mean photon number** is determined by the *expectation value* of the *photon number operator* $\hat{n} = a^\dagger\, a$,

$$\langle n\rangle_\alpha = \langle\alpha|\, \hat{n}\, |\alpha\rangle = |\alpha|^2. \tag{111}$$

Exc 21 Verify Eq. (111).

We can rewrite Eq. (110) as

$$p_\alpha(n) = \frac{\langle n\rangle_\alpha^n}{n!}\, e^{-\langle n\rangle_\alpha}, \tag{112}$$

which is the **Poisson distribution**.

Time Evolution

The photon **Hamiltonian** H is proportional to the photon **number operator** $\hat{n} = a^\dagger\, a$,

$$H = \omega\left(\hat{n} + \frac{1}{2}\right). \tag{113}$$

The **coherent states** (except $|\text{vac}\rangle$) are *not* energy eigenstates. ⇒ They evolve with time.

The **time-evolution operator** $U(t)$ is generated by H as

$$U(t) = e^{-i\, H\, t/\hbar} = e^{-\frac{i\,\omega\, t}{2}}\; e^{-i\,\omega\, t\,\hat{n}}. \tag{114}$$

Applying $U(t)$ to $|\alpha\rangle$:

$$U(t)\,|\alpha\rangle = e^{-\frac{i\,\omega\, t}{2}}\; \left|\alpha\, e^{-i\,\omega\, t}\right\rangle := e^{-\frac{i\,\omega\, t}{2}}\; |\alpha(t)\rangle. \tag{115}$$

Exc 22 Show Eq. (115).

So up to an overall phase factor $e^{-i\;\omega\;t/2}$ (originated from the zero-point energy), the parameter $\alpha = |\alpha|\, e^{i\,\varphi}$ evolves as

$$\alpha(t) = \alpha(0)\, e^{-i\,\omega\,t}, \tag{117}$$

such that

- the **amplitude** $|\alpha|$ remains the same,
- the **phase** $\varphi \to \varphi - \omega\, t$ will rotate with time t by the angular frequency ω.

According to Eq. (105), the electromagnetic fields expectation value will evolve as

$$\begin{aligned} \langle \alpha(t)|\, \boldsymbol{E}\, |\alpha(t)\rangle &= -(2\,\omega)^{1/2}\, \boldsymbol{e}\, |\alpha| \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega\, t), \\ \langle \alpha(t)|\, \boldsymbol{B}\, |\alpha(t)\rangle &= -(2/\omega)^{1/2}\, \boldsymbol{k}\times\boldsymbol{e}\, |\alpha| \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega\, t), \end{aligned} \tag{118}$$

describing the *dynamics* of the propagating electromagnetic wave throughout the *spacetime*.

The coherent state of electromagnetic field are quantum states that most closely resembles *classical light*.

- They *minimize* the **quantum fluctuation** in both *phase* and *amplitude* on top of the classical (average) behavior of wave, saturating their **uncertainty bound**.
- Large $|\alpha|$ ⇒ large average **number of photons** $\langle n\rangle_\alpha = |\alpha|^2$ in the coherent state ⇒ a *macroscopic* occupation of the same photon mode with *quantum coherence*, — making coherent states an ideal description of **laser light** (intense and coherent light).

Superradiant Light

Tavis-Cummings Model

The Tavis-Cummings model is an extension of the Jaynes-Cummings model, where a *single mode* **electromagnetic field** couples to a set of **two-level atoms** (i.e. *many* qubits), instead of a single two-level atom.

The Tavis-Cummings Hamiltonian is given by

$$H = \frac{\omega_0}{2} \sum_{i=1}^{N} \sigma_i^z + \omega \left(a^\dagger\, a + \frac{1}{2}\right) - g \sum_{i=1}^{N} \left(\sigma_i^-\, a^\dagger + \sigma_i^+\, a\right). \tag{119}$$

- N - total **number of atoms**, each indexed by $i = 1, 2, \ldots, N$.
- ω_0 - atom **excitation energy** (energy level splitting, transition frequency).
- ω - **photon energy** (mode frequency),
- g - light-matter **coupling strength**.

When $N = 1$, Eq. (119) reduces to the Jaynes-Cummings Hamiltonian in Eq. (92).

The full Hilbert space is a tensor product of the *atomic* and *photonic* degrees of freedom

$$\mathcal{H} = \underbrace{(\text{span}\{|g\rangle, |e\rangle\})^{\otimes N}}_{\text{atomic }(2^N\text{-dim})} \otimes \underbrace{\text{span}\{|0\rangle, |1\rangle, |2\rangle, \ldots\}}_{\text{photonic }(\infty\text{-dim})} \tag{120}$$

- Goal: find the **ground state** (lowest energy state) of H.
- Challenge: exact diagonalization is computationally difficult, given the huge (infinite) Hilbert space dimension.

U(1) Symmetry

The **number of excitations** N_{exc} (including both photons and atoms) is *conserved* in the Tavis-Cummings model.

$$N_{\text{exc}} = a^\dagger a + \sum_{i=1}^{N} \frac{\sigma_i^z + \mathbb{1}}{2}. \tag{121}$$

- **Symmetry ⇔ Conservation Law**: The excitation number conservation generates a **U(1) symmetry**, corresponding to the unitary operator (for any given U(1) rotation angle θ)

$$U(\theta) = e^{i\theta N_{\text{exc}}}. \tag{122}$$

- Under the U(1) **symmetry transformation**,

$$\begin{aligned} a &\to U(\theta)^\dagger\, a\, U(\theta) = e^{i\theta}\, a, \\ a^\dagger &\to U(\theta)^\dagger\, a^\dagger\, U(\theta) = e^{-i\theta}\, a^\dagger, \\ \sigma_i^\pm &\to U(\theta)^\dagger\, \sigma_i^\pm\, U(\theta) = e^{\mp i\theta}\, \sigma_i^\pm. \end{aligned} \tag{123}$$

Exc 23 Check Eq. (123).

Therefore, the Tavis-Cummings Hamiltonian H in Eq. (119) is invariant under the symmetry transformation, i.e. $\forall\, \theta : [H, U(\theta)] = 0$.

Mean-Field Approach

Idea: Replace the interacting *many-body* problem with several effective *single-body* (or *single-mode*) problem by approximating the effect of all other freedoms with an average (mean) field.

- **Variational Ansatz**: Propose a trial (variational) state that disentangle the atomic and photonic degrees of freedom.

$$|\Psi(\alpha)\rangle = |\psi(\alpha)\rangle_{\text{atom}}^{\otimes N} \otimes |\alpha\rangle_{\text{photon}}. \tag{125}$$

  - **Photons**: assumed to be in a *coherent state* with complex parameter α

$$|\alpha\rangle_{\text{photon}} = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{126}$$

Under U(1) symmetry transformation: $\alpha \to e^{i\theta}\,\alpha$.

- **Atoms**: assumed to be *identical product state* of

$$|\psi(\alpha)\rangle_{\text{atom}} = \psi_e(\alpha)\,|e\rangle + \psi_g(\alpha)\,|g\rangle, \tag{127}$$

which could also depend on the parameter α.

- **Objective**: Minimize the expectation value of the Tavis-Cummings Hamiltonian H with respect to the variational state.

$$\boxed{\min_{\alpha} E_{\text{MF}}(\alpha) := \langle\Psi(\alpha)|\, H\, |\Psi(\alpha)\rangle.} \tag{128}$$

Hope: the minimal energy state will be a good approximation of the true ground state within the *variational subspace*.

Mean-Field Energy

- **Photonic Expectation**: The photons are in a coherent state $|\alpha\rangle$,

$$\begin{aligned}\langle\alpha|\, a\, |\alpha\rangle &= \alpha,\\ \langle\alpha|\, a^\dagger\, |\alpha\rangle &= \alpha^*,\\ \langle\alpha|\, a^\dagger\, a\, |\alpha\rangle &= |\alpha|^2.\end{aligned} \tag{129}$$

Taking the expectation value of the photon part of H in Eq. (119),

$$\begin{aligned}\langle\alpha|\, H\, |\alpha\rangle &= \frac{\omega_0}{2}\sum_{i=1}^{N}\sigma_i^z + \omega\left(|\alpha|^2 + \frac{1}{2}\right) - g\sum_{i=1}^{N}(\sigma_i^-\,\alpha^* + \sigma_i^+\,\alpha)\\ &= \omega\left(|\alpha|^2 + \frac{1}{2}\right) + \sum_{i=1}^{N} H_i(\alpha),\end{aligned} \tag{130}$$

which has decoupled into an overall photon energy plus a sum of N identical effective atomic Hamiltonians $H_i(\alpha)$.

- **Atomic Expectation**: $H_i(\alpha)$ is the effective Hamiltonian for the ith atom on the photon coherent state background

$$H_i(\alpha) = \frac{\omega_0}{2}\,\sigma_i^z - g\,(\sigma_i^-\,\alpha^* + \sigma_i^+\,\alpha), \tag{131}$$

which can be represented as a 2×2 matrix in the $\{|e\rangle, |g\rangle\}$ basis,

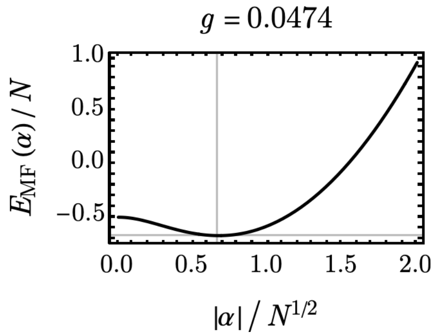
$$H_i(\alpha) \simeq \begin{pmatrix}\omega_0/2 & -g\,\alpha\\ -g\,\alpha^* & -\omega_0/2\end{pmatrix}, \tag{132}$$

whose minimal energy expectation value is given by the lowest energy eigenvalue:

$$\langle\psi(\alpha)|\, H_i(\alpha)\, |\psi(\alpha)\rangle = -\sqrt{\frac{\omega_0^2}{4} + g^2\,|\alpha|^2}\,. \tag{133}$$

Collecting Eq. (133) and Eq. (130), the **mean-field energy** $E_{\text{MF}}(\alpha)$ defined in Eq. (128) is given by

$$E_{\mathrm{MF}}(\alpha) = \omega\left(|\alpha|^2 + \frac{1}{2}\right) - N\sqrt{\frac{\omega_0^2}{4} + g^2\,|\alpha|^2}\,. \tag{134}$$



▫ Mean-Field Solutions

To find the optimal α that minimize $E_{\mathrm{MF}}(\alpha)$, solve for

$$\frac{\partial E_{\mathrm{MF}}(\alpha)}{\partial \alpha} = \left(2\,\omega - N\,g^2\left(\frac{\omega_0^2}{4} + g^2\,|\alpha|^2\right)^{-1/2}\right)\alpha^* = 0. \tag{135}$$

The solutions:

- **Trivial solution**: $\alpha = 0$ (always valid)
  - The mean-field energy reaches

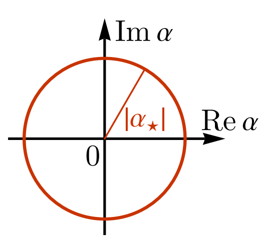
    $$E_{\mathrm{MF}}(0) = \frac{\omega}{2} - N\,\frac{\omega_0}{2}. \tag{136}$$

  - The variational state becomes

    $$|\Psi(0)\rangle = |g\rangle_{\mathrm{atom}}^{\otimes N} \otimes |0\rangle_{\mathrm{photon}}, \tag{137}$$

    describing: all **atoms** in *ground states* & **photon** *vacuum state*.
- **Non-trivial solution**: around the circle of

$$|\alpha_\star| = \frac{1}{2\,g\,\omega}\sqrt{g^4\,N^2 - (\omega\,\omega_0)^2} \xrightarrow{N \gg 1} \frac{N\,g}{2\,\omega}, \tag{138}$$



which are valid only if

$$\boxed{g > \sqrt{\frac{\omega\,\omega_0}{N}}\,.} \tag{139}$$

- The mean-field energy (always lower than $E_{\text{MF}}(0)$ as long as Eq. (139) holds)

$$E_{\text{MF}}(\alpha_\star) = \frac{\omega}{2} - \frac{1}{4}\left(\frac{g^2\,N^2}{\omega} + \frac{\omega\,\omega_0^2}{g^2}\right). \tag{140}$$

- The variation state is

$$|\Psi(\alpha_\star)\rangle = |\psi(\alpha_\star)\rangle_{\text{atom}}^{\otimes N} \otimes |\alpha_\star\rangle_{\text{photon}}. \tag{141}$$

In the large-N limit ($N \gg 1$): **photon** in a *coherent state* $|\alpha_\star\rangle$ with the average photon number (light intensity)

$$\langle n\rangle = |\alpha_\star|^2 \sim N^2, \tag{142}$$

and each **atom** in an equal-weight superposition of $|g\rangle$ and $|e\rangle$ with their relative phase locked to $\alpha_\star / |\alpha_\star|$

$$|\psi(\alpha_\star)\rangle \simeq \frac{1}{\sqrt{2}}\left(|g\rangle + \frac{\alpha_\star}{|\alpha_\star|}\,|e\rangle\right). \tag{143}$$

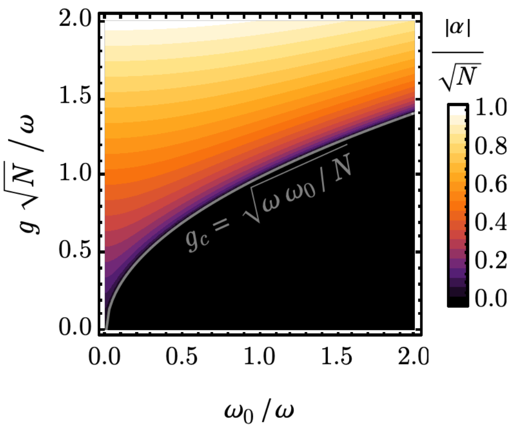
- **Spontaneous Symmetry Breaking**: Any choice of $\alpha_\star = |\alpha_\star|\,e^{i\,\theta}$ breaks the U(1) symmetry spontaneously, i.e. H respects the U(1) symmetry, but its (approximate) ground state $|\Psi(\alpha_\star)\rangle$ does not.

Superradiant Phase

The nontrivial solution $|\Psi(\alpha_\star)\rangle$ describes the **superradiant phase** of light, exhibiting key features:

- **Cooperative Radiation**: Many atoms radiate *coherently*, such that the emitted light intensity add *constructively*.
- **Strong Intensity**: *Macroscopic* photon occupation with $\langle n\rangle \sim N^2$ (in contrast to the linear N-scaling for independent spontaneous emission),
- **Phase Coherence**: Emitted photons are *phase-locked*, and the *collective* light-matter interaction stabilizes the phase of $\alpha_\star$, resulting in the *spontaneous breaking* of U(1) symmetry.

The superradiant transition **phase diagram**:



The superradiant phenomenon is closely related to **LASER** (Light Amplification by Stimulated Emission of Radiation). They share the mechanism of **stimulated emission**. However, laser is a steady state operating in a *driven*, *non-equilibrium* regime. It uses an *external pump* to maintain a *population inversion* of atoms, where stimulated emission overcomes photon losses, leading to continuous and coherent light output.