

130B Quantum Physics

Part 5. Phase and Gauge

Gauge Principles

■ Gauge Structure and Berry Phase

■ Phase Ambiguities

At its core, quantum mechanics is a **probability theory**. It postulate to *model* the probability density $p(\mathbf{x})$ by a *squared norm*

$$p(\mathbf{x}) = |\psi(\mathbf{x})|^2, \quad (1)$$

just to ensure the *positive semi-definite* property $p(\mathbf{x}) \geq 0$.

The wavefunction $\psi(\mathbf{x})$ itself serves as a **mathematical parameter** of the probability model, *not* a **physical observable**, and is therefore subject to some degree of ambiguity or redundancy.

- **Global phase ambiguity.** A *global* phase rotation of the wavefunction (where “global” means α does not depend on $\mathbf{x}$)

$$\psi(\mathbf{x}) \rightarrow e^{i\alpha} \psi(\mathbf{x})$$

 (2)

has no consequence on the expectation value $\langle O \rangle$ of any physical observable O in any case

$$\langle O \rangle = \int \psi^*(\mathbf{x}) O(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}') d^D \mathbf{x} d^D \mathbf{x}'. \quad (3)$$

Conclusion: quantum states $\in$ **projective Hilbert space**, where global phase is always *unphysical*.

- **Local phase ambiguity (Gauge redundancy).** We can push this idea further: if we restrict ourself to *diagonal* observables in the *position* basis, i.e., functions $f(\mathbf{x})$ that depends only on $\mathbf{x}$ (but not $\mathbf{p}$), then any *local* phase rotation

$$\psi(\mathbf{x}) \rightarrow e^{i\chi(\mathbf{x})} \psi(\mathbf{x})$$

 (4)

will leave all expectation values $\langle f(\mathbf{x}) \rangle$ invariant,

$$\langle f(\mathbf{x}) \rangle := \int f(\mathbf{x}) |\psi(\mathbf{x})|^2 d^D \mathbf{x} = \int f(\mathbf{x}) p(\mathbf{x}) d^D \mathbf{x} \quad (5)$$

since the probability density $p(\mathbf{x})$ is unchanged.

- If $p(\mathbf{x}) = |\psi(\mathbf{x})|^2$ was the only *physical* probability distribution to be modeled, any $\psi(\mathbf{x})$ related by *local* phase rotation Eq. (4) should be treated as *equivalent*.
- This represents a **gauge redundancy**: multiple *mathematical descriptions* (e.g. wavefunctions) describing the same *physical reality* (e.g. position distribution).

Quantum decoherence provides a deeper reason of why only *diagonal* observables are measurable.

- **Environmental monitoring**: The environment has a natural tendency to monitor the **local density** $p(\mathbf{x})$ of particles in the space.
 - Effectively performing *weak continuous measurement* of $\mathbf{x}$.
 - Inducing **decoherence** in the position basis: rapid decay of *off-diagonal* coherence $\rho(\mathbf{x}, \mathbf{x}')$ of the density matrix for $\mathbf{x} \neq \mathbf{x}'$.
- Consequence 1: **Dephasing noise**.
 - The measurement *randomizes* the **phase** of $\psi(\mathbf{x})$ at every position *independently*.
 - Relative phase between $\psi(\mathbf{x})$ and $\psi(\mathbf{x}')$ no longer comparable, allowing *local* phase rotation as Eq. (4) to become a *redundancy*.
- Consequence 2: **Gauge projection**.
 - The measurement *collapse* the system towards the **particle number** eigenstates, suppressing the *number fluctuations*.
 - Effectively imposing a *gauge constraint* (like Gauss law), that couples the particle to an *emergent gauge field*, allowing particles to interact with each other through *emergent gauge forces*.

■ Gauge Transformation

If the gauge freedom is an (emergent) *redundancy*, it should have no *physical consequence*. For example, it should not affect the **quantum dynamics** governed by the **Schrödinger equation**.

However, the standard Schrödinger equation is *not* invariant under the **gauge transformation**

$$\psi(\mathbf{x}, t) \rightarrow e^{i\chi(\mathbf{x}, t)} \psi(\mathbf{x}, t), \quad (6)$$

unless we modify it appropriately.

- Start from the free Schrödinger equation

$$i\hbar \partial_t \psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t), \quad (7)$$

under gauge transformation $\psi \rightarrow e^{i\chi} \psi$ in Eq. (6), the derivative operators picks up extra terms involving $\partial_t \chi$ and $\nabla \chi$,

$$i \hbar (\partial_t + i \partial_t \chi(\mathbf{x}, t)) \psi(\mathbf{x}, t) = -\frac{\hbar^2}{2m} (\nabla + i \nabla \chi(\mathbf{x}, t))^2 \psi(\mathbf{x}, t), \quad (8)$$

and the equation does *not* remain invariant.

Exc
1

Show that Eq. (7) becomes to Eq. (8) under gauge transformation.

- To restore the **gauge invariance**, we introduce **gauge fields**:

- **Scalar potential**: $\Phi(\mathbf{x}, t)$ - a scalar field in the spacetime
- **Vector potential**: $\mathbf{A}(\mathbf{x}, t)$ - a vector field in the spacetime

and replace derivatives by **covariant derivatives**:

$$\begin{aligned} \partial_t &\rightarrow D_t := \partial_t + \frac{i}{\hbar} \Phi(\mathbf{x}, t), \\ \nabla &\rightarrow \mathbf{D} := \nabla - \frac{i}{\hbar} \mathbf{A}(\mathbf{x}, t), \end{aligned} \quad (10)$$

Then Eq. (7) can be recast into the **gauge-invariant Schrödinger equation**

$$i \hbar D_t \psi(\mathbf{x}, t) = \frac{1}{2m} (-i \hbar \mathbf{D})^2 \psi(\mathbf{x}, t), \quad (11)$$

or more explicitly,

$$i \hbar \partial_t \psi(\mathbf{x}, t) = \left(\frac{1}{2m} (-i \hbar \nabla - \mathbf{A}(\mathbf{x}, t))^2 + \Phi(\mathbf{x}, t) \right) \psi(\mathbf{x}, t). \quad (12)$$

Under **gauge transformation**, the wavefunction ψ and the gauge fields $(\Phi, \mathbf{A})$ must transform together as

$$\begin{aligned} \psi(\mathbf{x}, t) &\rightarrow e^{i \chi(\mathbf{x}, t)} \psi(\mathbf{x}, t), \\ \Phi(\mathbf{x}, t) &\rightarrow \Phi(\mathbf{x}, t) - \hbar \partial_t \chi(\mathbf{x}, t), \\ \mathbf{A}(\mathbf{x}, t) &\rightarrow \mathbf{A}(\mathbf{x}, t) + \hbar \nabla \chi(\mathbf{x}, t), \end{aligned} \quad (13)$$

to ensure the covariance of the quantum dynamics.

■ Semiclassical Interpretation

In the WKB approximation, we write the wavefunction as

$$\psi = A e^{i S/\hbar}. \quad (14)$$

Plugging the WKB ansatz into the gauge-invariant Schrödinger equation Eq. (11) yields:

$$(-\partial_t S - \Phi) = \frac{1}{2m} (\nabla S - \mathbf{A})^2. \quad (15)$$

Given that the spacetime derivatives of the action S is associated to **energy** $E = -\partial_t S$ and

momentum $\mathbf{p} = \nabla S$, Eq. (15) can be written as

$$(E - \Phi) = \frac{1}{2m} (\mathbf{p} - \mathbf{A})^2. \quad (16)$$

This reveals the physical meaning of gauge fields:

- Scalar potential Φ : **potential energy**,
- Vector potential $\mathbf{A}$: **potential momentum**.

Note: Energy and Momentum each have *three* distinct forms

	Total = Kinetic + Potential	
Energy	$E = \frac{1}{2} m \dot{\mathbf{x}}^2 + \Phi$	
Momentum	$\mathbf{p} = m \dot{\mathbf{x}} + \mathbf{A}$	

(17)

**Exc
2**

Show that both equations in Eq. (17) are consistent with Eq. (16).

- **Total** (or **Canonical**): appear directly in *conservation laws* and determine the *action* accumulated in spacetime.
- **Kinetic**: directly linked to the particle's *motion* (velocity $\dot{\mathbf{x}}$).
- **Potential**: exist independently of particle motion, contributing even when the particle is *at rest* ($\dot{\mathbf{x}} = 0$), representing the *interaction* with the *background field* in the spacetime.

Question: What are their dynamical consequences?

Newton's 2nd law — the *force* $\mathbf{F}$ causes the *kinetic momentum* ($m \dot{\mathbf{x}}$) to change in time:

$$\mathbf{F} = \frac{d}{dt} (m \dot{\mathbf{x}}) = \frac{d\mathbf{p}}{dt} - \frac{d\mathbf{A}}{dt}, \quad (18)$$

in two distinct ways:

- $d\mathbf{p}/dt = \partial_t \mathbf{p} + \dot{\mathbf{x}} \cdot \nabla \mathbf{p}$, in which
 - $\nabla \mathbf{p} = 0$, as $\mathbf{x}$ and $\mathbf{p}$ are independent variables,
 - **Maxwell relation:** $\partial_t \nabla S = \nabla \partial_t S$ implies

$$\partial_t \mathbf{p} = -\nabla E = (\nabla \mathbf{A}) \cdot \dot{\mathbf{x}} - \nabla \Phi. \quad (19)$$

- $d\mathbf{A}/dt = \partial_t \mathbf{A} + \dot{\mathbf{x}} \cdot \nabla \mathbf{A}$.

Put together, $\mathbf{F}$ takes the form as the **Lorentz force** (on a charge $q = 1$ particle)

$$\begin{aligned} \mathbf{F} &= -\nabla \Phi - \partial_t \mathbf{A} + (\nabla \mathbf{A}) \cdot \dot{\mathbf{x}} - \dot{\mathbf{x}} \cdot \nabla \mathbf{A} \\ &= \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}, \end{aligned} \quad (20)$$

as long as we define

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi - \partial_t \mathbf{A}, \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{aligned} \tag{21}$$

Exc
3

Justify Eq. (19) and Eq. (20).

Obviously, $\mathbf{E}$ and $\mathbf{B}$ should be interpreted as **electric** and **magnetic** fields, allowing $\mathbf{F}$ to be consistently identified as the force exerted by the *electromagnetic field* on a *charged* particle.

Conclusion: Gauge fields (Φ , $\mathbf{A}$) are not merely mathematical constructs to maintain *gauge invariance*; they give rise to the physical *electromagnetic interactions* among quantum particles. Remarkably, the **electromagnetic forces** familiar from our everyday *classical* experience emerge profoundly from the **local phase ambiguity** of matter at the *quantum* level.

■ Math Interlude: Lorentz Vectors

It is more convenient to unify time and space, as well as energy and momentum.

- **Spacetime:** Introduce $x = (x^0, x^1, x^2, x^3)$ to denote the coordinate of a spacetime point,
 - $t = x^0$: time,
 - $\mathbf{x} = (x^1, x^2, x^3)$: space,
 and denote these components as x^μ ($\mu = 0, 1, 2, 3$) jointly. x is said to be a **Lorentz vector**.

- **Spacetime derivatives:** Partial derivatives in the spacetime are defined as

$$\partial_\mu f(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x^\mu}, \tag{22}$$

- $\partial_t = \partial_0$: temporal derivative,
- $\nabla = (\partial_1, \partial_2, \partial_3)$: spatial derivatives.
- **Energy-momentum:** Introduce $p = (p^0, p^1, p^2, p^3)$ to denote the energy and momentum,
 - $E = p^0 = -p_0$: energy,
 - $\mathbf{p} = (p^1, p^2, p^3) = (p_1, p_2, p_3)$: momentum.
- **Gauge field:** Introduce $A = (A^0, A^1, A^2, A^3)$ to denote the gauge field,
 - $\Phi = A^0 = -A_0$: scalar potential,
 - $\mathbf{A} = (A^1, A^2, A^3) = (A_1, A_2, A_3)$: vector potential.
- **Covariant derivatives:** Operators in Eq. (10) can now be unified as a single Lorentz vector operator

$$D_\mu = \partial_\mu - \frac{i}{\hbar} A_\mu, \tag{23}$$

- $D_t = D_0$: covariant temporal derivative,

- $\mathbf{D} = (D_1, D_2, D_3)$: covariant spatial derivative.

Raising and **lowering** the index of a Lorentz vector is done by the **Lorentz metric** $g_{\mu\nu}$ or $g^{\mu\nu}$,

$$a_\mu = g_{\mu\nu} a^\nu, \quad a^\mu = g^{\mu\nu} a_\nu, \quad (24)$$

where *repeated* indices are automatically summed over (**contracted**) following the **Einstein sum rule**. The Lorentz metric is given by

$$g_{\mu\nu} = g^{\mu\nu} = \text{diag}(-1, +1, +1, +1). \quad (25)$$

Rule of thumb: In **index contraction**, the *upper* index can only contract with the *lower* index and vice versa.

$$\begin{aligned} a^\mu b_\mu &\checkmark \text{ ok} \\ a^\mu b^\mu &\times \text{ no!} \\ a_\mu b_\mu &\times \text{ no!} \end{aligned}$$

More explicitly, the following expressions are all valid and equal

$$\begin{aligned} a^\mu b_\mu &= a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 \\ &= a^\mu g_{\mu\nu} b^\nu = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3 \\ &= a_\mu g^{\mu\nu} b_\nu = -a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3. \end{aligned} \quad (26)$$

■ Berry Phase

Berry phase is the *phase* accumulated by the wavefunction as a particle travels through the spacetime *adiabatically*.

- A processes is said to be **adiabatic**, if it happens *slowly* over a long time, i.e. the rates of change in physical observables tend to zero. In terms of the motion of a particle, it means the *velocity* of the particle is *almost zero* throughout the process:

$$\dot{\mathbf{x}} \rightarrow 0. \quad (27)$$

Eq. (27) is also called the *adiabatic limit*.

- In the adiabatic limit, the **action** of the particle is accumulated by the **potential** energy and momentum

$$\begin{aligned} -\partial_t S &= E = \Phi, \\ \nabla S &= \mathbf{p} = \mathbf{A}. \end{aligned} \quad (28)$$

as the *kinetic* energy and momentum *vanishes* when $\dot{\mathbf{x}} \rightarrow 0$.

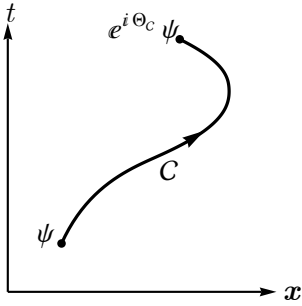
- Given that phase is related to action, the **Berry phase** that a particle accumulates along a spacetime trajectory C is given by the **path integral** that computes the accumulated action

$$\Theta_C = \frac{S_C}{\hbar} = \frac{1}{\hbar} \int_C -\Phi dt + \mathbf{A} \cdot d\mathbf{x} = \frac{1}{\hbar} \int_C A_\mu dx^\mu. \quad (29)$$

Exc 4 Justify Eq. (29).

Such that *adiabatically propagating* the **wave amplitude** ψ along trajectory C will acquire the **Berry phase** shift:

$$\psi \xrightarrow{C} e^{i\Theta_C} \psi = \exp\left(\frac{i}{\hbar} \int_C A_\mu dx^\mu\right) \psi. \quad (30)$$



- For infinitesimal transportation $C: x \rightarrow x + \delta x$,

$$\psi \rightarrow e^{\frac{i}{\hbar} A_\mu \delta x^\mu} \psi. \quad (31)$$

- Under gauge transformation

$$A_\mu(x) \rightarrow A_\mu(x) + \hbar \partial_\mu \chi(x), \quad (32)$$

- the Berry phase along an **open trajectory** C is *not* gauge invariant (hence not a physical observable):

$$\Theta_C \rightarrow \Theta_C + \chi(x_{\text{end}}) - \chi(x_{\text{start}}), \quad (33)$$

where $x_{\text{start}}, x_{\text{end}} = \partial C$ are the starting and ending point of C .

- the Berry phase around a **closed loop** (Wilson loop) is gauge invariant, and is a physical observable.

■ Gauge Field and Electromagnetism

■ Gauge Connection

Previously, we have introduced the *covariant derivative* D_μ to make the Schrödinger equation *gauge invariant*, but is there a deeper motivation behind this? In calculus, the **derivative** of a function tells us how much the function changes between nearby points

$$\partial_x f(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}. \quad (34)$$

But this assumes we can *compare* the values of the function at different points directly.

- **Problem:** If the wavefunction $\psi(x)$ has **local phase ambiguity**,

$$\psi(x) \rightarrow e^{i\chi(x)} \psi(x), \quad (35)$$

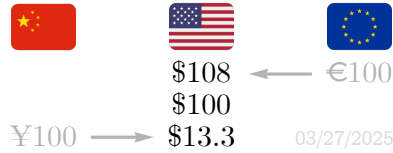
meaning that $\psi(x)$ at each point x is defined *up to a phase rotation*, there will be *no basis to compare* wavefunctions between *distinct points*.

☞ There is a similar issue in **finance**: you cannot directly compare currencies from different countries by their face values!

- Are the following amounts of money the same?



- You should first *move* (parallel transport) the money to the same place before comparing.



During the conversion, the money will be multiplied by the *exchange rate* (exponential gauge connection), e.g.

$$e^{0.076961} \text{€}100 = \$108, \quad e^{-2.01741} \text{¥}100 = \$13.3. \quad (36)$$

- **Solution:** Similarly, to define a meaningful derivative for the wavefunction, we have to introduce a **gauge connection** $A_\mu(x)$ to keep track of the *phase rotation* needed to transport $\psi(x + \delta x)$ back to the point x for comparison, for every point x along any direction μ .

This allows us to (re)define the **covariant derivative**:

$$D_\mu \psi(x) = \lim_{\delta x \rightarrow 0} \frac{e^{-\frac{i}{\hbar} A_\nu(x) \delta x^\nu} \psi(x + \delta x) - \psi(x)}{\delta x^\mu}. \quad (37)$$

- **Interpretation:** $\psi(x + \delta x)$ has accumulated a Berry phase of $\exp(\frac{i}{\hbar} A_\nu(x) \delta x^\nu)$ compare to $\psi(x)$ as the particle travels adiabatically. So when pulling $\psi(x + \delta x)$ back to the point x , this phase should be compensated by the opposite phase factor $\exp(-\frac{i}{\hbar} A_\nu(x) \delta x^\nu)$ before comparing.
- Expressed in terms of the usual partial derivative modified by the gauge connection,

$$D_\mu = \partial_\mu - \frac{i}{\hbar} A_\mu(x), \quad (38)$$

which exactly reproduces Eq. (23).

Exc
5

Show Eq. (38) follows from Eq. (37) by taking the limit.

- The covariant derivative commute with the **gauge transformation**

$$\begin{aligned}\psi(x) &\rightarrow e^{i\chi(x)}\psi(x), \\ A_\mu(x) &\rightarrow A_\mu(x) + \hbar\partial_\mu\chi(x),\end{aligned}\tag{39}$$

as adapted from Eq. (13).

- The **apparent** deformation of the wavefunction ψ from x to $x + \delta x$ can be expressed as

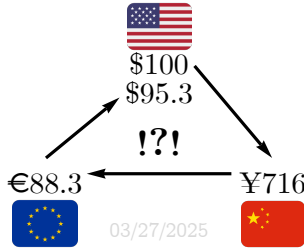
$$\psi(x + \delta x) = e^{\frac{i}{\hbar} A_\mu(x) \delta x^\mu} e^{\delta x^\mu D_\mu} \psi(x),\tag{40}$$

which contains two contributions

- the **intrinsic** deformation under **parallel transport** $\psi \rightarrow e^{\delta x^\mu D_\mu} \psi$, which is *generated* by the *covariant derivative* D_μ ,
- the **background** deformation in terms of the Berry phase $\psi \rightarrow e^{(i/\hbar) A_\mu(x) \delta x^\mu} \psi$ accumulated along the *gauge connection* A_μ .

■ Gauge Curvature

Exchanging currencies in cycles typically results in a loss. Why?



Because the global foreign exchange market is not *flat* — the mismatch around a *closed loop* is a measure of **curvature**.

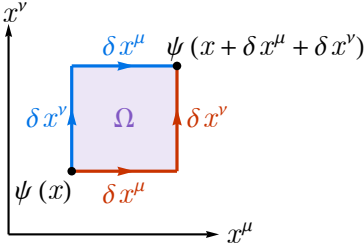
In gauge theory, the **gauge curvature** $F_{\mu\nu}$ measures the adiabatic *action* accumulated *per area* when transporting the wavefunction around the *area boundary* in spacetime.

$$S = \oint_{\partial\Omega} A_\mu dx^\mu = \int_{\Omega} F_{\mu\nu} dx^\mu dx^\nu.\tag{41}$$

- **Operational definition:** Mathematically, the gauge curvature $F_{\mu\nu}$ is defined by the *commutator of covariant derivatives*

$$[D_\mu, D_\nu] \psi = -\frac{i}{\hbar} F_{\mu\nu} \psi,\tag{42}$$

which measures the amount of non-commutativity to transport the wavefunction along two distinct spacetime directions μ and ν .



Exc 6

Using Eq. (40), prove Eq. (42) by comparing the above two paths to transport $\psi(x)$ to $\psi(x + \delta x^\mu + \delta x^\nu)$.

- Physical meaning: In electromagnetism, $F_{\mu\nu}$ corresponds to the **electromagnetic field strength tensor**,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (43)$$

Exc 7

Derive Eq. (43) from Eq. (42).

- **Electric field:** $\mathbf{E} = (E^1, E^2, E^3)$, with $E^i := -F_{0i}$.

$$\mathbf{E} = -\nabla\Phi - \partial_t \mathbf{A}. \quad (44)$$

- **Magnetic field:** $\mathbf{B} = (B^1, B^2, B^3)$, with $B^i := \frac{1}{2} \epsilon^{ijk} F_{jk}$.

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (45)$$

Exc 8

Check that $\mathbf{E}$ and $\mathbf{B}$ are gauge invariant. Therefore, they are physical observables.

■ Charged Particle in Gauge Field

In quantum mechanics, the time-evolution is generated by the Hamiltonian operator $\hat{H}$.

- **Schrödinger picture:** *state* evolves in time, operator remains fixed.

$$i \hbar \partial_t \psi = \hat{H} \psi. \quad (46)$$

- **Heisenberg picture:** *operator* evolves in time, state remains fixed.

$$i \hbar \partial_t \hat{O} = [\hat{O}, \hat{H}]. \quad (47)$$

Note: Eq. (47) assumes $\hat{O}$ has no explicit time dependence, if not, its *explicit time derivative* will also contribute to the rate of change of $\hat{O}$.

Compare Eq. (46) with Eq. (12), we conclude that the Hamiltonian of the **gauge-invariant Schrödinger equation** is

$$\hat{H} = -\frac{\hbar^2}{2m} \mathbf{D}^2 + \Phi, \quad (48)$$

or more explicitly as

$$\hat{H} = H(\hat{\mathbf{x}}, \hat{\mathbf{p}}, t) = \frac{1}{2m} (\hat{\mathbf{p}} - \mathbf{A}(\hat{\mathbf{x}}, t))^2 + \Phi(\hat{\mathbf{x}}, t), \quad (49)$$

where

- $\hat{\mathbf{x}}$ is the **coordinate operator**.
- $\hat{\mathbf{p}} = -i\hbar \nabla$ is the **momentum operator**.
- They satisfy the **canonical commutation relation**

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \mathbb{1}. \quad (50)$$

**Exc
9**

Verify Eq. (50).

- $\hat{\Phi} = \Phi(\hat{\mathbf{x}}, t)$ and $\hat{\mathbf{A}} = \mathbf{A}(\hat{\mathbf{x}}, t)$ are **operator functions** of $\hat{\mathbf{x}}$, with *explicit* time t dependence.

Using the Heisenberg equation Eq. (47), we can compute time derivatives of the particle position operator $\hat{\mathbf{x}}$

- 1st order (velocity operator)

$$\partial_t \hat{\mathbf{x}} = \frac{1}{i\hbar} [\hat{\mathbf{x}}, \hat{H}] = \frac{\hat{\mathbf{p}} - \hat{\mathbf{A}}}{m}. \quad (51)$$

- 2nd order (acceleration operator)

$$\begin{aligned} \partial_t^2 \hat{\mathbf{x}} &= -\frac{1}{m} \partial_t \hat{\mathbf{A}} + \frac{1}{i\hbar} [\partial_t \hat{\mathbf{x}}, \hat{H}] \\ &= \frac{1}{m} \left(\hat{\mathbf{E}} + \frac{1}{2} (\partial_t \hat{\mathbf{x}} \times \hat{\mathbf{B}} - \hat{\mathbf{B}} \times \partial_t \hat{\mathbf{x}}) \right), \end{aligned} \quad (52)$$

where $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ operators are defined by

$$\begin{aligned} \hat{\mathbf{E}} &= -\nabla \hat{\Phi} - \partial_t \hat{\mathbf{A}}, \\ \hat{\mathbf{B}} &= \nabla \times \hat{\mathbf{A}}. \end{aligned} \quad (53)$$

**Exc
10**

Derive Eq. (51) and Eq. (52).

Eq. (52) describes the **quantum dynamics** of a *charged particle* in an *electromagnetic field* in the Heisenberg picture:

$$m \partial_t^2 \hat{\mathbf{x}} = \hat{\mathbf{E}} + \frac{1}{2} \left(\partial_t \hat{\mathbf{x}} \times \hat{\mathbf{B}} - \hat{\mathbf{B}} \times \partial_t \hat{\mathbf{x}} \right). \quad (54)$$

In contrast, the **classical dynamics** is described by

$$m \ddot{\mathbf{x}} = \mathbf{F} = \mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}, \quad (55)$$

where all quantities commute. In the quantum case, however, $\partial_t \hat{\mathbf{x}}$ and $\hat{\mathbf{B}}$ generally do *not commute*, so their cross product must be symmetrized as in Eq. (54).

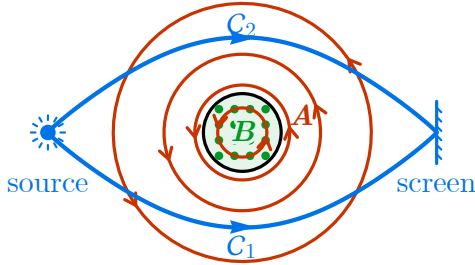
■ Aharonov-Bohm Effect

In quantum mechanics, the *motion* of a charged particle can be *influenced* by the **gauge fields** Φ and $\mathbf{A}$ through **quantum interference**, even in the *absence* of electromagnetic fields (i.e. $\mathbf{E} = \mathbf{B} = 0$) when there is no Lorentz force acting on the particle at all!

• Setup: Aharonov-Bohm Experiment

- **Physical Arrangement:** Consider a long, thin solenoid carrying a magnetic flux ϕ_B . Outside the solenoid, the magnetic field $\mathbf{B}$ is zero, but the vector potential $\mathbf{A}$ is nonzero. For any surface $\mathcal{S}$ that fully covers the solenoid, we have

$$\phi_B = \int_{\mathcal{S}} \mathbf{B} \cdot d\boldsymbol{\sigma} = \oint_{\partial \mathcal{S}} \mathbf{A} \cdot d\mathbf{l}. \quad (56)$$



- **Interferometry:** A beam of electrons is split into two paths that *encircle* the solenoid in opposite directions and then recombine to produce an *interference* pattern.
- **Key idea:** Even when $\mathbf{B} = 0$ outside the solenoid, the **vector potential** $\mathbf{A}$ influences the *phase* of the wavefunction.
- When an electron travels along a path $\mathcal{C}$, the wavefunction acquires a **Berry phase**:

$$\psi \xrightarrow{\mathcal{C}} \psi e^{i\Theta_{\mathcal{C}}} = \psi \exp\left(\frac{i q}{\hbar} \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{x}\right), \quad (57)$$

where $q = -e$ is recovered to represent the electron charge.

- The **phase difference** between the two paths is

$$\Delta\Theta = \Theta_{C_1} - \Theta_{C_2} = \frac{q}{\hbar} \left(\int_{C_1} \mathbf{A} \cdot d\mathbf{x} - \int_{C_2} \mathbf{A} \cdot d\mathbf{x} \right). \quad (58)$$

- By applying Stokes' theorem over the surface $\mathcal{S}$ enclosed by the loop $\mathcal{C} = C_1 - C_2 = \partial \mathcal{S}$,

$$\Delta\Theta = \frac{q}{\hbar} \int_S (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\sigma} = \frac{q\phi_B}{\hbar}, \quad (59)$$

where ϕ_B is the **magnetic flux** through $\mathcal{S}$ (which equals to the flux inside the solenoid as long as $\mathcal{S}$ covers the solenoid fully).

This phase shift manifests as a *shift* in the *interference fringes* when the two parts of the beam are *recombined*.

- **Application: Superconducting Quantum Interference Device (SQUID)**
 - A SQUID consists of a *superconducting loop* containing a *Josephson junction* (serving as the screen) and exploit quantum interference to detect extremely subtle changes in *magnetic flux* inside the loop.
 - By harnessing the *quantum-level sensitivity* of the Aharonov-Bohm (AB) effect, SQUIDS can measure magnetic fields as faint as $5 \times 10^{-18}\text{T}$ at *microscopic* scales.
 - SQUIDS also play a pivotal role in *quantum computing*, as an approach towards *superconducting qubits*.
- **Question: What Is Physical about Gauge Fields?**
 - **Gauge Potentials vs. Field Strengths:** Traditionally, one might think only the fields $\mathbf{E}$ and $\mathbf{B}$ are *physical* since they are *gauge invariant* and can be measured directly by *forces*. However, the AB effect shows that the potentials Φ and $\mathbf{A}$ also have direct physical consequences—they affect the phase of a quantum wavefunction.
 - **Holonomy and Berry Phase:** The Berry phase around any *closed loop* is gauge invariant, and should be *physical*. All such **closed-loop Berry phases** (aka. the **holonomies**) form the *complete* set of physical observables of a gauge theory. The AB phase is an example of a holonomy: the phase accumulated around a closed loop depends on the *curvature* (here, the magnetic flux ϕ_B) enclosed by the loop.

Uniform Magnetic Field

■ Classical Dynamics

■ Cyclotron Motion

The motion of a **charged particle** in the electromagnetic field is governed by the **Lorentz force**:

$m \ddot{\mathbf{x}} = \mathbf{F} = e (\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}).$

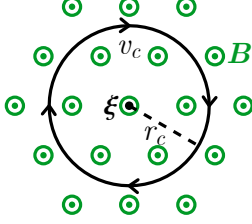
(60)

- m - particle **mass**,

- e - particle **charge**.

Consider the case: **uniform magnetic field only**,

$$\mathbf{E} = 0, \mathbf{B} = B \mathbf{e}^z. \quad (61)$$



Circular motion in x - y plane:

$$\begin{aligned} \dot{\mathbf{x}} &= v_c (\cos(\omega_c t) \mathbf{e}^x - \sin(\omega_c t) \mathbf{e}^y), \\ \mathbf{x} &= r_c (\sin(\omega_c t) \mathbf{e}^x + \cos(\omega_c t) \mathbf{e}^y) + \boldsymbol{\xi}, \end{aligned} \quad (62)$$

Exc 11

Demonstrate Eq. (62) by solving the equation of motion Eq. (60).

- ω_c - **cyclotron frequency**:

$$\omega_c = \frac{e B}{m}. \quad (68)$$

- v_c - **cyclotron velocity**, set by the initial velocity of the particle,
- r_c - **cyclotron radius**,

$$r_c = \frac{v_c}{\omega_c} = \frac{m v_c}{e B}. \quad (69)$$

- $\boldsymbol{\xi} = \xi_x \mathbf{e}^x + \xi_y \mathbf{e}^y$ - **guiding center** (the center of the cyclotron motion). It can be reconstructed from

$$\boldsymbol{\xi} = \mathbf{x} - \frac{r_c}{v_c} \mathbf{e}^z \times \dot{\mathbf{x}} = \mathbf{x} - \frac{1}{e B} \mathbf{e}^z \times \boldsymbol{\pi}, \quad (70)$$

where $\boldsymbol{\pi} = m \dot{\mathbf{x}}$ denotes the **kinetic momentum**.

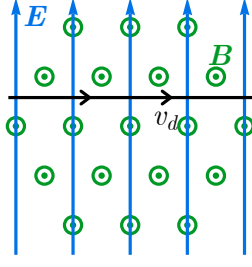
Exc 12

Verify Eq. (70).

■ Hall Effect

Consider the case: **uniform electric field perpendicular to uniform magnetic field**,

$$\mathbf{E} = E \mathbf{e}^y, \mathbf{B} = B \mathbf{e}^z. \quad (71)$$



The Lorentz force is balanced if

$$\dot{\mathbf{x}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = v_d \mathbf{e}^x. \quad (72)$$

- **Drift velocity** of charge:

$$v_d = \frac{E}{B}. \quad (73)$$

- Corresponding **current density**:

$$\mathbf{j} = n e v_d \mathbf{e}^x, \quad (74)$$

where

- n - carrier **density**, number of charge carriers per unit area.
- e - carrier **charge**.

**Exc
13**

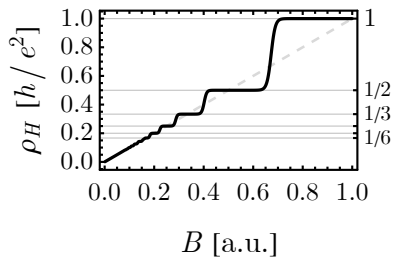
Justify Eq. (74).

- **Hall conductivity**:

$$\sigma_H := \frac{j_x}{E_y} = \frac{n e}{B}. \quad (75)$$

- **Hall resistivity**:

$$\rho_H := \frac{E_y}{j_x} = \frac{B}{n e}. \quad (76)$$



- Classical expectation: given *fixed* carrier (electron) density n , $\rho_H \propto B$ should scale *linearly* with B .
- **Quantum Hall effect:** when B is large enough, $\sigma_H = \rho_H^{-1}$ exhibits *steps* at *quantized* values:

$$\sigma_H = \frac{\nu e^2}{h} \quad (\nu = 1, 2, 3, \dots). \quad (77)$$

- h - Planck constant,
- e - electron charge.

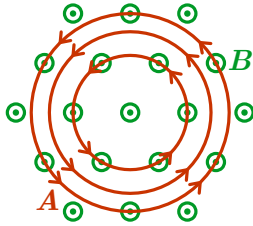
■ Landau Level Quantization

■ Gauge Field

Two-dimensional electron system in the x - y plane, with a **uniform magnetic field** perpendicular to the plane

$$\mathbf{B} = B \mathbf{e}^z = \nabla \times \mathbf{A}. \quad (78)$$

- The **vector potential** (gauge field) $\mathbf{A}$ circulates in the x - y plane,



$$\mathbf{A} = (A_x, A_y) = \frac{B}{2} (-y, x), \quad (79)$$

known as the **symmetric gauge**.

**Exc
14**

Verify Eq. (79) reproduces Eq. (78).

- However, the gauge choice is not unique. For example, the following gauge choice is also valid:

$$\mathbf{A} = (A_x, A_y) = (0, Bx),$$

known as the **Landau gauge**.

We will mostly work with the circular gauge Eq. (79), following Ref. [1].

[1] David Tong. The Quantum Hall Effect (TIFR Infosys Lectures), (2016).

■ Hamiltonian

Following Eq. (48), the Hamiltonian of the **gauge-invariant Schrödinger equation** reads

$$\hat{H} = -\frac{\hbar^2}{2m} \mathbf{D}^2 = -\frac{\hbar^2}{2m} \left(\nabla - \frac{ie}{\hbar} \hat{\mathbf{A}} \right)^2. \quad (80)$$

• Unit choice:

- charge $e = 1$,
- mass $m = 1$,
- Planck constant $\hbar = 1$.

Eq. (80) can be simplified to

$$\hat{H} = \frac{1}{2} (\hat{\mathbf{p}} - \hat{\mathbf{A}})^2 = \frac{1}{2} \hat{\boldsymbol{\pi}}^2. \quad (81)$$

• $\hat{\mathbf{p}} = -i \nabla = (-i \partial_x, -i \partial_y)$ - **canonical momentum** operator.

• **Canonical commutation** relation with **coordinate** operator $\hat{\mathbf{x}} = (\hat{x}, \hat{y})$:

$$[\hat{x}_i, \hat{p}_j] = i \quad (\text{for } i, j = x, y). \quad (82)$$

• $\hat{\mathbf{A}} = \mathbf{A}(\hat{\mathbf{x}}) = (\hat{A}_x, \hat{A}_y)$ - **potential momentum** operator (electromagnetic vector potential).

Under symmetric gauge Eq. (79),

$$\hat{A}_x = -\frac{B}{2} \hat{y}, \quad \hat{A}_y = \frac{B}{2} \hat{x}. \quad (83)$$

• $\hat{\boldsymbol{\pi}} = (\hat{\pi}_x, \hat{\pi}_y)$ - **kinetic momentum** operator,

$$\hat{\boldsymbol{\pi}} = \hat{\mathbf{p}} - \hat{\mathbf{A}} = -i \nabla - \mathbf{A}(\hat{\mathbf{x}}) = -i \mathbf{D}, \quad (84)$$

or, in terms of components

$$\begin{aligned} \hat{\pi}_x &= \hat{p}_x - \hat{A}_x = -i \partial_x - \hat{A}_x, \\ \hat{\pi}_y &= \hat{p}_y - \hat{A}_y = -i \partial_y - \hat{A}_y. \end{aligned} \quad (85)$$

• They satisfy the following commutation relation

$$[\hat{\pi}_x, \hat{\pi}_y] = i B, \quad (86)$$

which follows from the definition of **gauge curvature** in Eq. (42).

Exc
15

Verify Eq. (86).

• They also inherit the commutation relation with the coordinate operator,

$$[\hat{x}_i, \hat{\pi}_j] = i \quad (\text{for } i, j = x, y). \quad (87)$$

■ Guiding Center

Following Eq. (70), the **guiding center** operator $\hat{\xi} = (\xi_x, \xi_y)$ is defined as

$$\hat{\xi} = \hat{x} - \frac{1}{B} e^z \times \hat{\pi}. \quad (88)$$

- Under symmetric gauge Eq. (79),

$$\hat{\xi} = \frac{1}{B} (\hat{p} + \hat{A}) \times e^z, \quad (89)$$

Exc 16 | Verify Eq. (89).

or, in terms of components

$$\begin{aligned} \hat{\xi}_x &= \hat{x} + \frac{1}{B} \hat{\pi}_y = \frac{1}{B} (\hat{p}_y + \hat{A}_y), \\ \hat{\xi}_y &= \hat{y} - \frac{1}{B} \hat{\pi}_x = -\frac{1}{B} (\hat{p}_x + \hat{A}_x). \end{aligned} \quad (90)$$

- They satisfy the following commutation relation

$$[\hat{\xi}_x, \hat{\xi}_y] = \frac{1}{i B}. \quad (91)$$

Exc 17 | Prove Eq. (91).

- Guiding center and kinetic momentum operators commute.

$$[\hat{\xi}_i, \hat{\pi}_j] = 0 \quad (\text{for } i, j = x, y). \quad (92)$$

Exc 18 | Prove Eq. (92).

■ Annihilation and Creation Operators

Define two sets of annihilation and creation operators.

- **Cyclotron annihilation and creation operators:**

$$\hat{a} = \frac{1}{\sqrt{2 B}} (\hat{\pi}_x + i \hat{\pi}_y), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2 B}} (\hat{\pi}_x - i \hat{\pi}_y), \quad (93)$$

such that

$$\hat{a}^\dagger \hat{a} + \frac{1}{2} = \frac{1}{2B} \hat{\pi}^2 = \frac{1}{2B} (\hat{\mathbf{p}} - \hat{\mathbf{A}})^2. \quad (94)$$

Exc 19 | Verify Eq. (94).

- **Guiding center annihilation and creation operators:**

$$\hat{b} = \sqrt{\frac{B}{2}} (\hat{\xi}_x - i \hat{\xi}_y), \quad \hat{b}^\dagger = \sqrt{\frac{B}{2}} (\hat{\xi}_x + i \hat{\xi}_y), \quad (95)$$

such that

$$\hat{b}^\dagger \hat{b} + \frac{1}{2} = \frac{B}{2} \hat{\xi}^2 = \frac{1}{2B} (\hat{\mathbf{p}} + \hat{\mathbf{A}})^2. \quad (96)$$

Exc 20 | Verify Eq. (96).

They satisfy the following commutation relations

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1, \quad [\hat{b}, \hat{b}^\dagger] = 1, \\ [\hat{a}, \hat{b}] &= [\hat{a}, \hat{b}^\dagger] = [\hat{a}^\dagger, \hat{b}] = [\hat{a}^\dagger, \hat{b}^\dagger] = 0. \end{aligned} \quad (97)$$

Exc 21 | Prove Eq. (97).

- Implication: $\hat{a}$ and $\hat{b}$ are *annihilation operators* for two *independent harmonic oscillator* degrees of freedom.
- $\hat{a}^\dagger \hat{a}$ and $\hat{b}^\dagger \hat{b}$ are commuting **number operators**, and can be diagonalized *simultaneously*. Their eigenvalues correspond to two *separate* sets of **quantum numbers**, denoted as n and m respectively:

$$\begin{aligned} \hat{a}^\dagger \hat{a} |n, m\rangle &= n |n, m\rangle, \\ \hat{b}^\dagger \hat{b} |n, m\rangle &= m |n, m\rangle, \end{aligned} \quad (98)$$

$$n, m = 0, 1, 2, \dots \in \mathbb{N}.$$

■ Landau Levels

The system has two important physical observables:

- **Hamiltonian:** using Eq. (94), Eq. (81) can be written as

$$\hat{H} = \frac{1}{2} \hat{\pi}^2 = B \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \quad (99)$$

- **Angular momentum:**

$$\hat{L}_z = (\hat{\mathbf{x}} \times \hat{\mathbf{p}}) \cdot \mathbf{e}^z = \hat{b}^\dagger \hat{b} - \hat{a}^\dagger \hat{a}. \quad (100)$$

**Exc
22**

Verify Eq. (100) under symmetric gauge.

Obviously, $[\hat{H}, \hat{L}_z] = 0$, i.e. $\hat{H}$ and $\hat{L}_z$ can be simultaneously diagonalized.

- Their common eigenstates are $|n, m\rangle$:

$$\begin{aligned} \hat{H} |n, m\rangle &= B \left(n + \frac{1}{2} \right) |n, m\rangle, \\ \hat{L}_z |n, m\rangle &= (m - n) |n, m\rangle, \end{aligned} \quad (101)$$

which are labeled by two quantum numbers:

- n : **Landau level** index (energy level index),
- m : **angular momentum** index (degeneracy within a Landau level).
- The energy levels are quantized

$$E_n = B \left(n + \frac{1}{2} \right), \quad (102)$$

with $n = 0, 1, 2, \dots \in \mathbb{N}$.

- After restoring the energy unit,

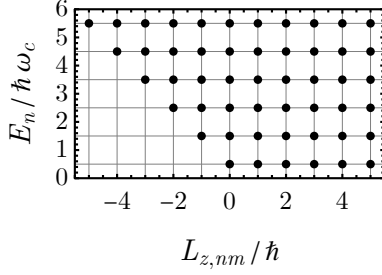
$$E_n = \frac{\hbar q B}{m} \left(n + \frac{1}{2} \right) = \hbar \omega_c \left(n + \frac{1}{2} \right). \quad (103)$$

where $\omega_c = q B / m$ is the **cyclotron frequency**.

- Each level is called a **Landau level**. The $n = 0$ level is called the **lowest Landau level** (LLL).
- The angular momentum is also quantized. After restoring the unit,

$$L_{z, nm} = \hbar (m - n). \quad (104)$$

Eigenstates $|n, m\rangle$ arranged by energy v.s. angular momentum.



• **Landau level degeneracy:**

- In an *infinite* system, each Landau level is *infinitely* degenerated.
 - **Argument:** The quantum number $m = 0, 1, 2, \dots \in \mathbb{N}$ is unbounded. $\Rightarrow$ Infinitely many orthogonal states $|n, m\rangle$ within the same energy level n .
- In a realistic system of *finite* size, the Landau level degeneracy becomes *finite*, and is determined by the *total magnetic flux* measured in units of the *flux quantum*.
 - Consider electrons confined within a disk of radius R . $\Rightarrow$ The *guiding center* radius ξ must satisfy $\xi \lesssim R$.
 - This puts a constraint on the operator

$$\hat{b}^\dagger \hat{b} + \frac{1}{2} = \frac{B}{2} \hat{\xi}^2 \lesssim \frac{B R^2}{2}, \quad (105)$$

in terms of its eigenvalues. $\Rightarrow m \lesssim B R^2 / 2$, restoring the physical units:

$$m \lesssim \frac{e B R^2}{2 \hbar}. \quad (106)$$

- A **flux quantum** is the amount of magnetic flux ϕ_0 that induces a 2π **Berry phase** for an electron braiding around it.

$$\text{phase} = \frac{\text{action}}{\hbar} = \frac{e \phi_0}{\hbar} = 2\pi, \quad (107)$$

meaning that

$$\phi_0 = \frac{h}{e}. \quad (108)$$

- Then Eq. (106) can be written as

$$m \lesssim \frac{\phi_B}{\phi_0}, \quad (109)$$

where $\phi_B = \pi R^2 B$ is the **total magnetic flux** through the disk.

Thus the Landau level degeneracy is set by ϕ_B / ϕ_0 — the conclusion generalizes to any shape of the system.

■ Complex Coordinate

Instead of using $\mathbf{x} = (x, y)$ to coordinate the position of the particle, it is more convenient to introduce:

- the **complex coordinate**

$$z = \sqrt{\frac{B}{2}} (x + i y), \quad \bar{z} = \sqrt{\frac{B}{2}} (x - i y), \quad (110)$$

- and the **complex derivative**

$$\partial_z = \frac{1}{\sqrt{2} B} (\partial_x - i \partial_y), \quad \partial_{\bar{z}} = \frac{1}{\sqrt{2} B} (\partial_x + i \partial_y). \quad (111)$$

Exc 23 | Derive Eq. (111) from Eq. (110).

Using the complex notation, the creation and annihilation operators can be represented as

$$\begin{aligned} \hat{a} &= -i \left(\frac{1}{2} z + \partial_{\bar{z}} \right), \quad \hat{a}^\dagger = i \left(\frac{1}{2} \bar{z} - \partial_z \right); \\ \hat{b} &= \frac{1}{2} \bar{z} + \partial_z, \quad \hat{b}^\dagger = \frac{1}{2} z - \partial_{\bar{z}}. \end{aligned} \quad (112)$$

Exc 24 | Verify Eq. (112).

■ Wave Functions

$|0,0\rangle$ is the **vacuum state** for both $\hat{a}$ and $\hat{b}$ bosons, defined by the condition

$$\hat{a} |0,0\rangle = \hat{b} |0,0\rangle = 0. \quad (113)$$

so the vacuum state *wave function* $\psi_{0,0}(z, \bar{z})$ should satisfy

$$\left(\frac{1}{2} z + \partial_{\bar{z}} \right) \psi_{0,0}(z, \bar{z}) = \left(\frac{1}{2} \bar{z} + \partial_z \right) \psi_{0,0}(z, \bar{z}) = 0. \quad (114)$$

- Consider the ansatz

$$\psi_{0,0}(z, \bar{z}) = f(z, \bar{z}) e^{-\bar{z} z/2}, \quad (115)$$

Eq. (114) implies

$$\partial_z f(z, \bar{z}) = \partial_{\bar{z}} f(z, \bar{z}) = 0, \quad (116)$$

Exc 25 | Derive Eq. (116).

meaning that $f(z, \bar{z})$ must be *independent* of both z and $\bar{z}$, i.e. it is a constant function, i.e.

$$f(z, \bar{z}) = \text{const.} \quad (117)$$

- Therefore, the vacuum state wave function reads

$$\psi_{0,0}(z, \bar{z}) = \frac{1}{\sqrt{\pi}} e^{-\bar{z} z/2}, \quad (118)$$

which is normalized to ensure $\int d z d \bar{z} |\psi_{0,0}(z, \bar{z})|^2 = 1$.

Any other state $|n, m\rangle$ can be raised from the vacuum state $|0, 0\rangle$ by applying creation operators

$$|n, m\rangle = \frac{1}{\sqrt{n! m!}} (\hat{a}^\dagger)^n (\hat{b}^\dagger)^m |0, 0\rangle, \quad (119)$$

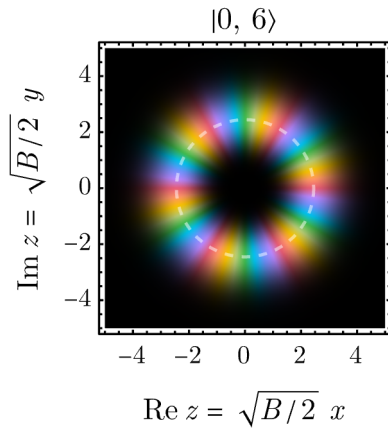
for $n, m = 0, 1, 2, \dots$

- **Lowest Landau level (LLL):** The wave functions with $n = 0$ are

$$\psi_{0,m}(z, \bar{z}) = \frac{1}{\sqrt{\pi m!}} z^m e^{-\bar{z} z/2}. \quad (120)$$

Exc 26 | Derive Eq. (120).

The wave functions has circular symmetry, taking the **ring shape**.



- Probability density

$$|\psi_{0,m}(z, \bar{z})|^2 = \frac{1}{\pi m!} (\bar{z} z)^m e^{-\bar{z} z}, \quad (121)$$

which distributes around the radius $|z| = \sqrt{m}$.

**Exc
27**

Estimate the radius $|z|$ that maximizes the probability.

- **Semi-classical argument:** The factor $z^m = \sqrt{B/2} r^m e^{i m \theta}$ implies that the particle accumulates $2 \pi m$ *Berry phase* when travelling around a circle of *radius* r .

- To ensure that the wave function is single-valued, m must be an integer, i.e.

$$2 \pi m \in 2 \pi \mathbb{Z} \Rightarrow m \in \mathbb{Z}. \quad (122)$$

- This corresponds to a *canonical momentum* of

$$p = \frac{2 \pi m}{2 \pi r} = \frac{m}{r}. \quad (123)$$

- States in the LLL should *minimize* the energy $E = (\mathbf{p} - \mathbf{A})^2 / 2$, which can be achieved if

$$p = A = \frac{\pi r^2 B}{2 \pi r} = \frac{r B}{2}. \quad (124)$$

- Eq. (123) and Eq. (124) together sets the optimal radius

$$\frac{m}{r} = p = \frac{r B}{2} \Rightarrow r = \sqrt{\frac{2 m}{B}}, \quad (125)$$

or, in terms of z variable, $|z| = \sqrt{B/2} r = \sqrt{m}$ (with $m \in \mathbb{Z}$).

- Any linear combination of $|0, m\rangle$ is still a state in the LLL, which takes the general form of

$$\sum_m c_m \psi_{0,m}(z, \bar{z}) = \frac{1}{\sqrt{\pi m!}} \left(\sum_m c_m z^m \right) e^{-\bar{z} z/2} = f(z) e^{-\bar{z} z/2}. \quad (126)$$

Conclusion: LLL wavefunctions are **holomorphic functions** of z , multiplied by a **Gaussian envelope**.

- More generally, for higher Landau levels, the wave functions

$$\psi_{n,m}(z, \bar{z}) = \frac{1}{\sqrt{\pi n! m!}} ((\bar{z} - \partial_z)^n z^m) e^{-\bar{z} z/2}, \quad (127)$$

can be written in terms of *associated Laguerre polynomials*.

■ Quantum Hall Effect

■ Filling Landau Levels

For a **single electron** confined to a *two-dimensional plane* under a *uniform perpendicular*

magnetic field, the energy eigenvalues are quantized into **Landau levels**:

$$E_n = \hbar \omega_c \left(n + \frac{1}{2} \right) \quad (n = 0, 1, 2, \dots). \quad (128)$$

- **Level spacing** (energy gap): $\hbar \omega_c$, where $\omega_c = e B / m$ is the *cyclotron frequency*.
- **Level degeneracy**: Given the total magnetic flux ϕ_B through the plane, degeneracy is

$$N_\phi := \frac{\phi_B}{\phi_0} = \frac{B A}{\phi_0}, \quad (129)$$

where

- A - total area of the system,
- $\phi_0 = h / e$ - the *magnetic flux quantum*.

Many-body system: In real materials, there is not just one single electron, but **many electrons** interacting with each other — the problem become *many-body* in nature.

- The key feature of electrons is their **fermionic statistics**.
 $\Rightarrow$ **Pauli exclusion principle**: no two electrons can occupy the same quantum state.
- **Filling up energy levels**: Due to Pauli exclusion, electrons will fill up *available quantum states* starting from the lowest energy states.
- **Filling fraction** ν is the (fractional) number of Landau levels that will be filled up,

$$\nu = \frac{N}{N_\phi} = \frac{n h}{e B}.$$

(130)

- N - *total number* of electron in the system,
- $n = N / A$ - **electron density**, i.e. the number of electrons per area.

Conversely, Eq. (130) allows us to express n in terms of ν ,

$$n = \frac{\nu e}{h} B. \quad (131)$$

The phenomenology of **integer quantum Hall effect** is that the Hall conductivity σ_H takes quantized values (recall Eq. (77))

$$\sigma_H = \frac{n e}{B} = \frac{\nu e^2}{h}, \quad (132)$$

at $\nu = 1, 2, 3, \dots$, i.e. when the **filling fraction** $\nu \in \mathbb{N}$ is an integer, corresponding to the situation where

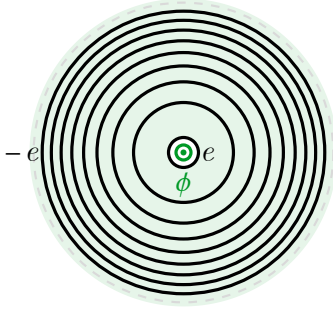
- the lowest ν **Landau levels** are completely *filled*,
- and all higher levels are completely *empty*.

This leads to an **incompressible quantum state**: electrons cannot change states without a large energy cost (the Landau level spacing $\hbar \omega_c$).

Message: The *quantized* Hall conductance originates from fully filling *discrete* Landau levels. — Why does each filled Landau level contribute to exactly one unit of σ_H ?

■ Charge Pumping

Consider a *annular geometry* that allows us to probe the Hall effect via **radial charge pumping**.



- In addition to a uniform perpendicular magnetic field B , we thread an extra *magnetic flux* $\phi(t)$ through the central hole, which *slowly* increases from 0 to ϕ_0 over a *long* time T ,

$$\phi(t) = \phi_0 \frac{t}{T}, \quad (133)$$

- $\phi_0 = h/e$ - flux quantum,
- $T \gg 1/\omega_c$ - *adiabatic* evolution time. ($t: 0 \rightarrow T$).
- Due to the **Aharonov-Bohm effect**, each quantum state $|n, m\rangle$ will be affected — its wave function radius r_m will be deformed adiabatically,

$$r_m = \sqrt{\frac{\phi_0}{\pi B}} \sqrt{m - t/T}. \quad (134)$$

**Exc
28**

Derive Eq. (134).

Under the flux insertion,

$$\begin{array}{llll} t: & 0 & \rightarrow & T \\ \phi(t): & 0 & \rightarrow & \phi_0 \\ r_m \propto \sqrt{m} & \rightarrow & \sqrt{m-1} & \end{array}, \quad (136)$$

the radius of every orbit shrinks from $\sqrt{m}$ to $\sqrt{m-1} \Rightarrow$ pumping *one charge* e from the *outer* edge to the *inner* edge.

- During this process,
 - the radial current density:

$$j_r = \frac{1}{2\pi r} \frac{e}{T}, \quad (137)$$

- the circular electric field induced by the changing flux:

$$E_\theta = \frac{1}{2\pi r} \frac{d\phi(t)}{dt} = \frac{1}{2\pi r} \frac{\phi_0}{T}, \quad (138)$$

Hall conductivity for each fully filled Landau level:

$$\sigma_H = \frac{j_r}{E_\theta} = \frac{e}{\phi_0} = \frac{e^2}{h}. \quad (139)$$

Conclusion: *quantized* transport of charge in response to *quantized* flux insertion $\Rightarrow$ *quantized* Hall conductance.

■ Linear Response Theory

The **Hall conductivity** σ_H characterizes how the **current density** $\mathbf{j}$ (observable) *responds* to the **electric field** $\mathbf{E}$ (perturbation).

- Current density operator for *each electron* (assuming $m = e = \hbar = 1$) reads

$$\hat{\mathbf{j}} = \frac{B}{N_\phi} \hat{\boldsymbol{\pi}}. \quad (140)$$

**Exc
29**

Justify Eq. (140) based on Eq. (74).

- The *expectation value* of **current density** in the *system* is given by

$$\begin{aligned} \langle \hat{\mathbf{j}} \rangle &= \sum_{n,m \in \text{occ}} \langle n,m | \hat{\mathbf{j}} | n,m \rangle \\ &= \frac{B}{N_\phi} \sum_{n,m \in \text{occ}} \langle n,m | \hat{\boldsymbol{\pi}} | n,m \rangle, \end{aligned} \quad (141)$$

where $\sum_{n,m \in \text{occ}}$ is to sum over all lowest $|n,m\rangle$ states that are *occupied* by the electron.

- Based on Eq. (93), the **kinetic momentum** operator $\hat{\boldsymbol{\pi}}$ can be expressed as

$$\hat{\pi}_x = \sqrt{\frac{B}{2}} (\hat{a}^\dagger + \hat{a}), \quad \hat{\pi}_y = \sqrt{\frac{B}{2}} i (\hat{a}^\dagger - \hat{a}). \quad (142)$$

As expected, the current density $\langle \hat{\mathbf{j}} \rangle = 0$ vanishes on the many-body ground state, in the absence of perturbation.

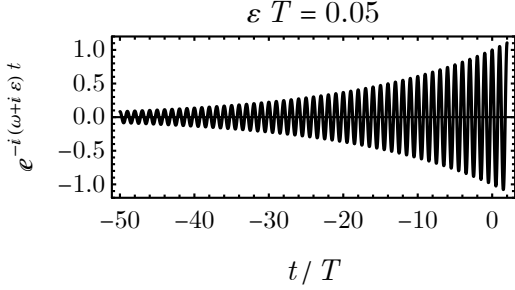
- Applying a (weak) **electric field** $\mathbf{E}$ to the system amounts to perturbing the **vector potential** $\mathbf{A}$ by a *time-dependent* perturbation $\delta \mathbf{A}(t)$,

$$\mathbf{A} \rightarrow \mathbf{A} + \delta \mathbf{A}(t), \quad (143)$$

with $\mathbf{E}(t) = -\partial_t \delta \mathbf{A}(t)$, according to Eq. (44).

- Assume that $\mathbf{E}(t) = \mathbf{E} e^{-i(\omega+i0_+)t}$ takes an oscillatory form with frequency ω , and is adiabatically turned on from the infinite past, then

$$\delta \mathbf{A}(t) = \int_{-\infty}^t dt' \mathbf{E}(t') = -\frac{\mathbf{E}}{i\omega} e^{-i(\omega+i0_+)t}. \quad (144)$$



- This means the **Hamiltonian operator** will be perturbed by

$$\hat{H} \rightarrow \hat{H} + \delta \hat{H}(t), \quad (145)$$

with the perturbation given by

$$\delta \hat{H}(t) = \frac{\partial \hat{H}}{\partial \mathbf{A}} \cdot \delta \mathbf{A}(t) \quad (146)$$

to the leading order of $\delta \mathbf{A}(t)$.

- Given that $\hat{\pi} = \hat{\mathbf{p}} - \mathbf{A}$ and

$$\hat{H} = \frac{1}{2} \hat{\pi}^2 = \frac{1}{2} (\hat{\mathbf{p}} - \mathbf{A})^2, \quad (147)$$

we have

$$\frac{\partial \hat{H}}{\partial \mathbf{A}} = -(\hat{\mathbf{p}} - \mathbf{A}) = -\hat{\pi}. \quad (148)$$

Substitute Eq. (148) into Eq. (146), the *time-dependent perturbation* Hamiltonian reads

$$\delta \hat{H}(t) = -\hat{\pi} \cdot \delta \mathbf{A}(t) = \frac{1}{i\omega} \mathbf{E} \cdot \hat{\pi} e^{-i(\omega+i0_+)t}. \quad (149)$$

The time-dependent perturbation problem can be solved by the Green's function approach.

- **Dressed Green's function** (unitary time-evolution operator) can be computed from the Dyson series to the leading order

$$\hat{G}(t, -\infty) = \hat{G}_0(t, -\infty) + (-i) \int_{-\infty}^t dt' \hat{G}_0(t, t') \delta \hat{H}(t') \hat{G}_0(t', -\infty) + \dots, \quad (150)$$

where the **bare Green's function** is defined by

$$\hat{G}_0(t, t') = \sum_{n,m} |n,m\rangle e^{-i E_n(t-t')} \langle n,m|, \quad (151)$$

with $E_n = B(n + 1/2)$ being the **Landau level energy** (see Eq. (102)).

- Every state will evolve in time by

$$|n,m\rangle \rightarrow \hat{G}(t, -\infty) |n,m\rangle. \quad (152)$$

As a result, by Eq. (141), the current density expectation value $\langle \mathbf{j} \rangle$ evolves as

$$\begin{aligned} \langle \mathbf{j}(t) \rangle &= \frac{B}{N_\phi} \sum_{n,m \in \text{occ}} \langle n,m| \hat{G}(-\infty, t) \hat{\boldsymbol{\pi}} \hat{G}(t, -\infty) |n,m\rangle \\ &= \frac{B}{N_\phi} \sum_{n,m \in \text{occ}} \langle n,m| \left(\hat{\boldsymbol{\pi}}(t) + \frac{1}{\omega} \int_{-\infty}^t dt' e^{-i(\omega+i0_+)t'} [\mathbf{E} \cdot \hat{\boldsymbol{\pi}}(t'), \hat{\boldsymbol{\pi}}(t)] + \dots \right) |n,m\rangle. \end{aligned} \quad (153)$$

**Exc
30**

Derive Eq. (153).

where we have introduced

$$\hat{\boldsymbol{\pi}}(t) := \hat{G}_0(-\infty, t) \hat{\boldsymbol{\pi}} \hat{G}_0(t, -\infty). \quad (154)$$

- The first term in Eq. (153) is the current density in the *absence* of an electric field, which *vanishes*.

$$\sum_{n,m \in \text{occ}} \langle n,m| \hat{\boldsymbol{\pi}}(t) |n,m\rangle = 0. \quad (155)$$

**Exc
31**

Show Eq. (155).

- The second term in Eq. (153) describes the **linear response** of the current density under the electric field, which takes the form of

$$\langle \mathbf{j}(t) \rangle = \mathbf{E} \cdot \boldsymbol{\sigma}(\omega) e^{-i(\omega+i0_+)t}, \quad (156)$$

with the **conductivity matrix** $\sigma_{ji}(\omega)$ given by

$$\sigma_{ji}(\omega) = \frac{B}{\omega N_\phi} \int_{-\infty}^0 dt e^{-i(\omega+i0_+)t} \sum_{n,m \in \text{occ}} \langle n,m| [\hat{\pi}_j(t), \hat{\pi}_i(0)] |n,m\rangle. \quad (157)$$

**Exc
32**

Derive Eq. (156) and Eq. (157).

Eq. (156) implies that when the applied electric field $\mathbf{E}(t) = \mathbf{E} e^{-i(\omega+i0_+)t}$ oscillates at frequency ω , the induced current $\langle \mathbf{j}(t) \rangle$ also oscillates at the *same* frequency. — A feature of *linear* response.

- The **Hall conductivity** corresponds to the off-diagonal component $\sigma_{xy}(\omega)$ of the conductivity matrix. In the DC limit $\omega \rightarrow 0$, it is given by the **Kubo formula**:

$$\sigma_H := \lim_{\omega \rightarrow 0} \sigma_{xy}(\omega) = \frac{B}{i N_\phi} \sum_{n,m \in \text{occ}} \sum_{n' \neq n} \frac{\langle n,m | \hat{\pi}_x | n',m \rangle \langle n',m | \hat{\pi}_y | n,m \rangle - h.c.}{(E_{n'} - E_n)^2}. \quad (158)$$

**Exc
33**

Derive Eq. (158).

- Using Eq. (142) to represent the kinetic momentum operator $\hat{\pi}$ as annihilation and creation operators, and given that

$$\begin{aligned} \hat{a} |n,m\rangle &= \sqrt{n} |n-1,m\rangle, \\ \hat{a}^\dagger |n,m\rangle &= \sqrt{n+1} |n+1,m\rangle, \end{aligned} \quad (159)$$

the Hall conductivity σ_H in Eq. (158) reduces to

$$\sigma_H = \frac{1}{N_\phi} \sum_{n,m \in \text{occ}} 1. \quad (160)$$

**Exc
34**

Derive Eq. (160).

- Each occupied state contributes $1 / N_\phi$ to σ_H (in unit of the conductance quantum e^2 / h).
- Each Landau level is N_ϕ -fold degenerated. $\Rightarrow$ Fully filling each Landau level produces exactly one unit of σ_H .

Spin and Monopole

■ Classical Spin

■ Angular Momentum Decomposition

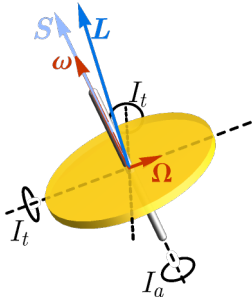
The classical motion of a **spinning top** is governed by

$$\tau = \frac{d\mathbf{L}}{dt}, \quad (161)$$

where

- τ - **torque** exerted on the top,
- $\mathbf{L}$ - total **angular momentum** of the top, decomposed into components *parallel* $I_a \boldsymbol{\omega}$ and *perpendicular* $I_t \boldsymbol{\Omega}$ to the spinning axis

$$\mathbf{L} = I_a \boldsymbol{\omega} + I_t \boldsymbol{\Omega}, \quad (162)$$



- I_a - **axial moment of inertia** (about the spinning axis),
- I_t - **transverse moment of inertia** (about either of the two equivalent transverse axes)
- $\boldsymbol{\omega}$ - **axial angular velocity** (along the spinning axis)
- $\boldsymbol{\Omega}$ - **transverse angular velocity**, describing the instantaneous rotation rate of the spinning axis itself.

■ Dynamics of Spinning Axis

Substitute Eq. (162) into Eq. (161),

$$\boldsymbol{\tau} = I_a \frac{d\boldsymbol{\omega}}{dt} + I_t \frac{d\boldsymbol{\Omega}}{dt}. \quad (163)$$

- **Transverse Torque Assumption:** Assume $\boldsymbol{\tau}$ has no component along the spinning axis, so the *magnitude* of $\boldsymbol{\omega}$ remains constant. Only its *direction* changes due to the rotation of the spinning axis:

$$\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{\omega}, \quad (164)$$

Eq. (163) becomes

$$\boldsymbol{\tau} = I_a \boldsymbol{\Omega} \times \boldsymbol{\omega} + I_t \frac{d\boldsymbol{\Omega}}{dt}. \quad (165)$$

We are mainly interested in the *motion* of the **spinning axis**, represented by the *unit vector*

$$\mathbf{n} = \frac{\boldsymbol{\omega}}{\omega}.$$

(166)

Similar to Eq. (164), $\mathbf{n}$ also gets rotated by $\boldsymbol{\Omega}$ as

$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\Omega} \times \mathbf{n}, \quad (167)$$

from which $\boldsymbol{\Omega}$ can be “solved” and expressed as

$$\boldsymbol{\Omega} = \boldsymbol{n} \times \frac{d\boldsymbol{n}}{dt}. \quad (168)$$

Exc 35 | Derive Eq. (168) from Eq. (167).

Substitute Eq. (168) into Eq. (165), and cross product with $\boldsymbol{n}$ from right on both sides, we obtain

$$I_t \left(\frac{d^2 \boldsymbol{n}}{dt^2} \right)_{\perp} = \boldsymbol{\tau} \times \boldsymbol{n} - \frac{d\boldsymbol{n}}{dt} \times \boldsymbol{S}, \quad (169)$$

where

- $\boldsymbol{S} := I_a \boldsymbol{\omega} = S \boldsymbol{n}$ - **spin angular momentum** (*parallel* to the spinning axis),
- $(\ddot{\boldsymbol{n}})_{\perp} = \ddot{\boldsymbol{n}} - (\ddot{\boldsymbol{n}} \cdot \boldsymbol{n}) \boldsymbol{n}$ - component of acceleration $\ddot{\boldsymbol{n}}$ in the tangent plane.

Exc 36 | Derive Eq. (169).

■ Magnetic Monopole

■ Electromagnetic Analogy

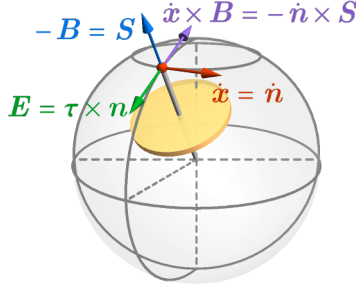
The motion of the spinning axis $\boldsymbol{n}$ (**spin dynamics**) can be interpreted as the motion of a *charged particle* on a *unit sphere* with an *magnetic monopole* inside (**charge dynamics**).

- **Analogy:** Compare the spin dynamics in Eq. (169) and the charge dynamics in Eq. (55)

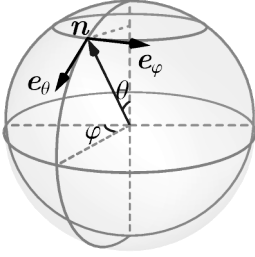
$$\begin{aligned} \text{spin: } I_t (\ddot{\boldsymbol{n}})_{\perp} &= \boldsymbol{\tau} \times \boldsymbol{n} - \dot{\boldsymbol{n}} \times \boldsymbol{S} \\ \text{charge: } m \ddot{\boldsymbol{x}} &= \boldsymbol{E} + \dot{\boldsymbol{x}} \times \boldsymbol{B} \end{aligned} \quad (170)$$

Spin dynamics	Charge dynamics
Spin orientation : $\boldsymbol{n}$	Charge position : $\boldsymbol{x}$
Moment of inertia : I_t	Mass (interia) : m
Torque-induced force : $\boldsymbol{\tau} \times \boldsymbol{n}$	Electric field : $\boldsymbol{E}$
Spin angular momentum : $-\boldsymbol{S}$	Magnetic field : $\boldsymbol{B}$

- **Similarity:** Just as in electromagnetism, where the Lorentz force *deflects* a charge moving in a magnetic field, the spin-induced term $-\dot{\boldsymbol{n}} \times \boldsymbol{S}$ generates *precession* of the spinning axis.



- **Difference:** The constraint to the *sphere* makes the coordinate system *non-Euclidean* (curved), in which ∇ operator is defined differently. [2]



- **Spherical coordinate:** parametrize the spin axis

$$\mathbf{n} = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta), \quad (171)$$

by the *polar angle* $\theta \in [0, \pi]$ and the *azimuthal angle* $\varphi \in [0, 2\pi)$.

- **Unit vectors:**

$$\begin{aligned} \mathbf{e}^\theta &= (\cos \varphi \cos \theta, \sin \varphi \cos \theta, -\sin \theta), \\ \mathbf{e}^\varphi &= (-\sin \varphi, \cos \varphi, 0). \end{aligned} \quad (172)$$

- **Surface gradient:** $\nabla_\perp \Phi$ for a scalar field Φ ,

$$\nabla_\perp \Phi = \mathbf{e}^\theta \partial_\theta \Phi + \mathbf{e}^\varphi \frac{1}{\sin \theta} \partial_\varphi \Phi. \quad (173)$$

- **Surface curl:** $\nabla_\perp \times \mathbf{A}$ for a vector field $\mathbf{A} = A_\theta \mathbf{e}^\theta + A_\varphi \mathbf{e}^\varphi$,

$$\nabla_\perp \times \mathbf{A} = \frac{1}{\sin \theta} (\partial_\theta (A_\varphi \sin \theta) - \partial_\varphi A_\theta) \mathbf{n}. \quad (174)$$

[2] Wikipedia. Del in cylindrical and spherical coordinates.

□ Differential Geometry for Vector Calculus

In **vector calculus**, we often compute *gradients*, *divergences*, *curls*, and integrals over curves, surfaces, or volumes. While powerful, these operations depend heavily on the *coordinate system* and *dimension*. **Differential geometry** provide a more *geometric* and *coordinate-free* language for calculus. They unify many familiar operations and extend naturally to curved

spaces (manifolds).

- **Differential Forms:** differential form is the basic object in differential geometry, it is an object that you can *integrate*:
- A **0-form** is just a *scalar* field $f(x)$, which can be integrated (evaluated) on a **point** x .
- A **1-form** is a *vector* field (*line* element), like $\omega(x) = \omega_i(x) dx^i$, something you can integrate along a **curve**.
- A **2-form** is an *tensor* field (oriented *surface* element), e.g. $\sigma_{ij}(x) dx^i \wedge dx^j$, integrable over a **surface**.

In general:

- A **k -form** is something you integrate over a k -dimensional submanifold.
- The **wedge product** $\wedge$ defines an oriented, antisymmetric product between forms:

$$dx^i \wedge dx^j = -dx^j \wedge dx^i. \quad (175)$$

- **Metric:** The metric defines how distance is measured on the manifold.

$$ds^2 = g_{ij}(x) dx^i dx^j, \quad (176)$$

where

- ds^2 is the squared infinitesimal distance element,
- $g_{ij}(x)$ are the components of the metric tensor, forming a symmetric positive-definite matrix.
- dx^i are the differential 1-form basis (cotangent basis)
- **Exterior Derivative:** the exterior derivative d acts on differential forms to produce forms of higher degree. Given $f = f_I dx^I$,

$$df = \partial_i f_I dx^i \wedge dx^I. \quad (177)$$

- $d^2 = 0$: the exterior derivative of an exterior derivative is always zero.
- **Stoke's Theorem:**

$$\int_{\partial M} \omega = \int_M d\omega. \quad (178)$$

- **Hodge Dual:** On an n -dimensional oriented manifold, the Hodge star operator $\star$ maps k -forms to $(n - k)$ -forms.

$$\star(dx^{i_1} \wedge \dots \wedge dx^{i_k}) = \frac{|\det[g_{ij}]|^{1/2}}{(n - k)!} g^{i_1 j_1} \dots g^{i_k j_k} \epsilon_{j_1 \dots j_n} dx^{j_{k+1}} \wedge \dots \wedge dx^{j_n}, \quad (179)$$

where

- $\epsilon_{j_1 \dots j_n}$ is the **Levi-Civita symbol**,

- $g^{ij} = \langle dx^i, dx^j \rangle$ is the **inverse metric**, which tells how differential forms dx^i and dx^j are “angled” with respect to each other.

Exterior derivative and Hodge dual enable us to represent **vector calculus operators** in terms of differential forms in a unified manner.

Vector Calculus Differential Forms	
grad (∇f)	df
curl $(\nabla \times \omega)$	$\star d\omega$
div $(\nabla \cdot \omega)$	$\star d\star\omega$
Laplacian $(\nabla^2 f)$	$\star d\star df$

(180)

See Refs. [3] for more details of the above concepts.

Application in spherical coordinates on S^2 .

- **Metric:** the distance element is given by

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2, \quad (181)$$

which implies

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2} \theta \end{pmatrix}, \quad (182)$$

and $|\det[g_{ij}]|^{1/2} = \sin \theta$.

- **Unit (co)vectors:** orthonormal basis of 1-forms

$\begin{aligned} e^\theta &\leftrightarrow \sqrt{g_{\theta\theta}} d\theta = d\theta, \\ e^\varphi &\leftrightarrow \sqrt{g_{\varphi\varphi}} d\varphi = \sin \theta d\varphi. \end{aligned}$

(183)

This enables us to represent any *vector* field as *1-form* field, such as

$$\mathbf{A} = A_\theta e^\theta + A_\varphi e^\varphi \leftrightarrow A = A_\theta d\theta + A_\varphi \sin \theta d\varphi. \quad (184)$$

- **Gradient:** given a scalar field Φ ,

$\text{grad } \Phi = e^\theta \partial_\theta \Phi + e^\varphi \frac{1}{\sin \theta} \partial_\varphi \Phi.$
--

(185)

**Exc
37**

Derive Eq. (185) using differential geometry approach.

- **Curl:** given a vector field $\mathbf{A} = A_\theta e^\theta + A_\varphi e^\varphi$,

$\text{curl } \mathbf{A} = \frac{1}{\sin \theta} (\partial_\theta (A_\varphi \sin \theta) - \partial_\varphi A_\theta) \mathbf{n}.$

(186)

Exc 38 | Derive Eq. (186) using differential geometry approach.

- **Divergence:** given a vector field $\mathbf{A} = A_\theta \mathbf{e}^\theta + A_\varphi \mathbf{e}^\varphi$,

$$\operatorname{div} \mathbf{A} = \frac{1}{\sin \theta} (\partial_\theta (A_\theta \sin \theta) + \partial_\varphi A_\varphi). \quad (187)$$

Exc 39 | Derive Eq. (187) using differential geometry approach.

- **Laplacian:** given a scalar field ψ ,

$$\nabla^2 \psi = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \psi) + \frac{1}{\sin^2 \theta} \partial_\varphi^2 \psi. \quad (188)$$

Exc 40 | Derive Eq. (188) using differential geometry approach.

[3] Vincent Bouchard. MATH 315: Calculus IV (University of Alberta).

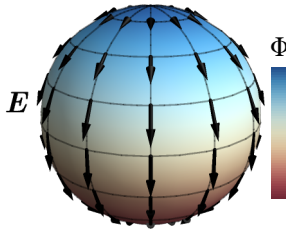
■ Effective Gauge Field

Introduce the effective gauge field $(\Phi, \mathbf{A})$ on the sphere, such that

- Effective **electric field**:

$$\mathbf{E}(\mathbf{n}) = -\nabla_\perp \Phi(\mathbf{n}). \quad (189)$$

In this way, $\mathbf{E}$ is guaranteed to lie in the tangent plane, the same as the torque-induced force $\boldsymbol{\tau} \times \mathbf{n}$.



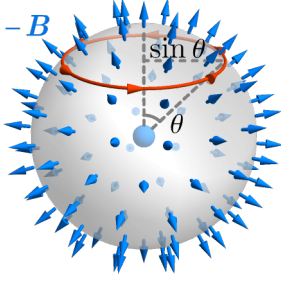
Example: a spinning top in a uniform gravity field $\Phi(\mathbf{n}) \propto n_z = \cos \theta$,

$$\mathbf{E} = -\nabla_\perp \Phi = \mathbf{e}_\theta \sin \theta. \quad (190)$$

- Effective **magnetic field**:

$$\mathbf{B}(\mathbf{n}) = -S \mathbf{n} = \nabla_\perp \times \mathbf{A}(\mathbf{n}). \quad (191)$$

where $S = I_a \omega$ is the spin angular momentum (magnitude). The magnetic field points towards the origin, where there is effective an **magnetic monopole**.



One common choice of vector potential $\mathbf{A}(\mathbf{n})$ to produce such magnetic field is the **Wu-Yang monopole potential** [4]

$$\mathbf{A}(\mathbf{n}) = S \frac{\cos \theta - 1}{\sin \theta} \mathbf{e}_\varphi \quad (192)$$

**Exc
41**

Verify that $\mathbf{A}(\mathbf{n})$ given in Eq. (192) satisfies Eq. (191).

- As a simple justification, along a *latitude loop* at the polar angle θ , the loop integral of the vector potential is

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int S \frac{\cos \theta - 1}{\sin \theta} \sin \theta d\varphi = 2\pi S (\cos \theta - 1), \quad (193)$$

which indeed equals to the *magnetic flux* through the *spherical cap* from the north pole down to angle θ

$$\int \mathbf{B} \cdot d\boldsymbol{\sigma} = -S \Omega(\theta) = -2\pi S (1 - \cos \theta), \quad (194)$$

where $\Omega(\theta) = 2\pi (1 - \cos \theta)$ denotes the *solid angle* of the cap.

- However, as $\theta \rightarrow \pi$ near the south pole, $A_\varphi \rightarrow \infty$ diverges. What is going wrong?

[4] T. T. Wu, C. N. Yang. Dirac monopole without strings: monopole harmonics. Nuclear Physics B107 365-380 (1976).

■ Quantization of Spin (or Monopole)

The divergence of A_φ is due to the requirement that an *infinitesimal latitude loop* near the south pole ($\theta \rightarrow \pi$) must accumulate a finite amount of **Berry phase** set by the *total magnetic flux* through the sphere

$$\Theta = \frac{1}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l} = \frac{2\pi}{\hbar} \sin \theta A_\varphi = \frac{4\pi B}{\hbar} = -\frac{4\pi S}{\hbar}. \quad (195)$$

The divergence can not be avoid, unless ... Θ is actually *equivalent* to 0, i.e.

$$\Theta = 2\pi n, \quad (196)$$

with $n \in \mathbb{Z}$.

- Therefore, it is only possible to avoid singular assignment of $\mathbf{A}(\mathbf{n})$ if the **spin angular momentum** S is *quantized* to

$$S = \frac{\hbar}{2} n, \quad (197)$$

with $n = 0, 1, 2, \dots$

- This is also a statement about the **magnetic monopole**, that the **total magnetic flux** emitted by a magnetic monopole must be quantized to

$$\phi_B = 4\pi B = -2\pi\hbar n. \quad (198)$$

Mathematically, the singularity is avoided by using two overlapping coordinate patches (north and south hemispheres) with smooth gauge fields on each.

- On the **northern hemisphere** ($\theta \in [0, \pi/2]$, excluding the south pole):

$$\mathbf{A}_N(\mathbf{n}) = S \frac{\cos\theta - 1}{\sin\theta} \mathbf{e}_\varphi. \quad (199)$$

- On the **southern hemisphere** ($\theta \in [\pi/2, \pi]$, excluding the north pole):

$$\mathbf{A}_S(\mathbf{n}) = S \frac{\cos\theta + 1}{\sin\theta} \mathbf{e}_\varphi. \quad (200)$$

- On the **equator** ($\theta = \pi/2$) where both patches overlap, the two gauge potentials are related by a **gauge transformation**:

$$\mathbf{A}_N(\varphi) - \mathbf{A}_S(\varphi) = \hbar \mathbf{e}_\varphi \partial_\varphi \chi(\varphi), \quad (201)$$

with $\chi(\varphi) = -2(S/\hbar)\varphi$.

Since φ and $\varphi + 2\pi$ correspond to the *same* point on the equator, the gauge transformation $\psi(\varphi) \rightarrow e^{i\chi(\varphi)}\psi(\varphi)$ is only consistent if

$$\begin{aligned} e^{i\chi(\varphi)} &= e^{i\chi(\varphi+2\pi)} \\ \Rightarrow \exp(-i2(S/\hbar)\varphi) &= \exp(-i2(S/\hbar)(\varphi+2\pi)) \end{aligned} \quad (202)$$

which requires

$$\frac{2S}{\hbar} \in \mathbb{Z}, \quad (203)$$

reproducing the **spin quantization condition** in Eq. (197).

■ Quantum Spin

■ Hamiltonian

Consider the **spherical symmetric** case, where there is no external scalar potential

$\Phi(\mathbf{n}) = 0$. Similar to Eq. (48), the quantum dynamics of a spin is described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2 I_t} \mathbf{D}_\perp^2 + \frac{S^2}{2 I_a}. \quad (204)$$

- $I_t = m$ - the *transverse moment of inertia* of the spin,
- $\mathbf{D}_\perp$ - **surface covariant derivative**, along tangent directions on the sphere,

$$\mathbf{D}_\perp = \nabla_\perp - \frac{i}{\hbar} \mathbf{A}, \quad (205)$$

- $\nabla_\perp = e^\theta \partial_\theta + e^\varphi \frac{1}{\sin \theta} \partial_\varphi$ - **surface gradient**,
- $\mathbf{A} = A_\theta e^\theta + A_\varphi e^\varphi$ - **gauge connection** (vector potential) on the sphere. Take the Wu-Yang monopole potential:

$$A_\theta = 0, \quad A_\varphi = S \frac{\cos \theta - 1}{\sin \theta}, \quad (206)$$

with the quantization condition ($n \in \mathbb{N}$),

$$S = \hbar s = \hbar \frac{n}{2}, \quad (207)$$

where $s = n/2$ is introduced as the **spin quantum number** (quantized to half integers), also characterizing the *monopole strength*.

In spherical coordinate, the Hamiltonian acts on the wave function $\psi(\theta, \varphi)$ as

$$\mathbf{D}_\perp^2 \psi = \frac{1}{\sin \theta} \left(\partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin \theta} (\partial_\varphi + i s (1 - \cos \theta))^2 \right) \psi. \quad (208)$$

Exc
42

Derive Eq. (208) using differential geometry approach.

■ Schrödinger Equation

Solve the stationary **Schrödinger equation**:

$$\hat{H} \psi(\theta, \varphi) = E \psi(\theta, \varphi). \quad (209)$$

- Separation of variables: let

$$\psi(\theta, \varphi) = e^{i(m-s)\varphi} \psi(\theta), \quad (210)$$

Eq. (209) takes the form of the generalized associated Legendre differential equation

$$\left(\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) - \frac{1}{\sin^2 \theta} (m - s \cos \theta)^2 + \lambda \right) \psi(\theta) = 0 \quad (211)$$

where the eigenvalue λ is related to the energy E by

$$E = \frac{\hbar^2}{2 I_t} \lambda + \frac{\hbar^2}{2 I_a} s^2. \quad (212)$$

**Exc
43**

Justify Eq. (211) and Eq. (212).

- The equation Eq. (211) can be solved by

$$\psi(\theta) = (1 - \cos \theta)^{\frac{s-m}{2}} (1 + \cos \theta)^{\frac{s+m}{2}} P_{l-s}^{(s-m, s+m)}(\cos \theta), \quad (214)$$

where

- $P_n^{(a,b)}(x)$ denotes the Jacobi polynomial,

$$P_n^{(a,b)}(x) = 2^{-n} \sum_{k=0}^n \frac{(n+a)!}{k! (n+a-k)!} \frac{(n+b)!}{(n-k)! (b+k)!} (x-1)^{n-k} (x+1)^k. \quad (215)$$

which is well-defined for $x \in [-1, 1]$ if $n, n+a, n+b \in \mathbb{N}$, implying the following quantization conditions:

- $l = s, s+1, s+2, \dots,$
- $m = -l, -l+1, \dots, l-1, l.$
- The corresponding eigenvalue λ is

$$\lambda = l(l+1) - s^2. \quad (216)$$

**Exc
44**

Verify Eq. (214) and Eq. (216).

■ Monopole Harmonics

Put together, define the **monopole harmonics** $Y_{slm}(\theta, \varphi)$ function

$$Y_{slm}(\theta, \varphi) = \mathcal{N} e^{i(m-s)\varphi} (1 - \cos \theta)^{\frac{s-m}{2}} (1 + \cos \theta)^{\frac{s+m}{2}} P_{l-s}^{(s-m, s+m)}(\cos \theta). \quad (220)$$

where $P_n^{(a,b)}$ is the Jacobi polynomial, and $\mathcal{N}$ is the *normalization* constant to ensure $\int_0^{2\pi} d\varphi \int_0^\pi d\theta |Y_{slm}(\theta, \varphi)|^2 \sin \theta = 1$.

- The eigen wavefunction of $\hat{H}$ is given by

$$\psi_{slm}(\theta, \varphi) = Y_{slm}(\theta, \varphi). \quad (221)$$

- The corresponding eigen energy is

$$E_{slm} = \frac{\hbar^2}{2 I_t} (l(l+1) - s^2) + \frac{\hbar^2}{2 I_a} s^2. \quad (222)$$

In the *isotropic* limit ($I_a = I_t = I$), the eigen energy only depends on the quantum number l ,

$$E_{slm} = \frac{\hbar^2}{2I} l(l+1). \quad (223)$$

- The **quantum numbers** s, l, m take values in
 - $s = 0, 1/2, 1, 3/2, \dots \in \mathbb{N}/2$, with $S = \hbar s \Rightarrow$ sets the monopole strength (i.e. total **magnetic flux** through the unit sphere).
 - $l = s, s+1, s+2, \dots \in s + \mathbb{N} \Rightarrow$ labels the **Landau levels** on the sphere.
 - $m = -l, -l+1, \dots, l-1, l \Rightarrow$ labels the degenerated states within each Landau level $\Rightarrow$ Landau level- l has the **degeneracy** $(2l+1)$.

■ Angular Momentum

The total **angular momentum operator** is defined as

$$\hat{\mathbf{L}} = \mathbf{n} \times (-i\hbar \mathbf{D}_\perp) + \mathbf{S}. \quad (224)$$

where

- $\mathbf{S} = S \mathbf{n} = \hbar s \mathbf{n}$ is the **spin angular momentum** (*axial* component).
- $\mathbf{n} \times (-i\hbar \mathbf{D}_\perp)$ is the **orbital angular momentum** (*transverse* component) associated with the spinning axis precession, which is given by the *cross product* between:
 - $\mathbf{n}$ - axis **coordinate** (on the sphere),
 - $-i\hbar \mathbf{D}_\perp$ - axis **kinetic momentum** (in the tangent plane).

Explicitly,

$$\hat{\mathbf{L}} = \hbar \left(\mathbf{e}^\varphi (-i\partial_\theta) - \mathbf{e}^\theta \frac{1}{\sin \theta} ((-i\partial_\varphi) + s(1 - \cos \theta)) + s \mathbf{n} \right). \quad (225)$$

**Exc
45**

Derive Eq. (225).

According to Eq. (171) and Eq. (172),

$$\begin{aligned} \mathbf{n} &= \cos \varphi \sin \theta \mathbf{e}^x + \sin \varphi \sin \theta \mathbf{e}^y + \cos \theta \mathbf{e}^z, \\ \mathbf{e}^\theta &= \cos \varphi \cos \theta \mathbf{e}^x + \sin \varphi \cos \theta \mathbf{e}^y - \sin \theta \mathbf{e}^z, \\ \mathbf{e}^\varphi &= -\sin \varphi \mathbf{e}^x + \cos \varphi \mathbf{e}^y. \end{aligned} \quad (226)$$

$\hat{\mathbf{L}}$ in Eq. (225) can be decomposed in Cartesian coordinate system as

$$\hat{\mathbf{L}} = \hat{L}_x \mathbf{e}^x + \hat{L}_y \mathbf{e}^y + \hat{L}_z \mathbf{e}^z, \quad (227)$$

with

$$\hat{L}_x = \hbar \left(\sin \varphi (i\partial_\theta) + \cos \varphi \left(\cot \theta (i\partial_\varphi) + s \frac{1 - \cos \theta}{\sin \theta} \right) \right),$$

$$\hat{L}_y = \hbar \left(-\cos \varphi (i \partial_\theta) + \sin \varphi \left(\cot \theta (i \partial_\varphi) + s \frac{1 - \cos \theta}{\sin \theta} \right) \right),$$

$$\hat{L}_z = \hbar (-i \partial_\varphi + s).$$

**Exc
46**

Derive Eq. (228).

They satisfy the following commutation relations:

$$\begin{aligned} [\hat{L}_x, \hat{L}_y] &= i \hbar \hat{L}_z, \\ [\hat{L}_y, \hat{L}_z] &= i \hbar \hat{L}_x, \\ [\hat{L}_z, \hat{L}_x] &= i \hbar \hat{L}_y, \end{aligned} \quad (229)$$

which could be summarized as $\hat{\mathbf{L}} \times \hat{\mathbf{L}} = i \hbar \hat{\mathbf{L}}$ in vector form. This is the defining property of any angular momentum operator.

**Exc
47**

Verify Eq. (229).

■ Squared Angular Momentum

The **squared angular momentum operator** is generally defined as

$$\hat{\mathbf{L}}^2 := \hat{\mathbf{L}} \cdot \hat{\mathbf{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2. \quad (230)$$

- For our case of $\hat{\mathbf{L}}$ in Eq. (224), $\hat{\mathbf{L}}^2$ can be explicitly written out

$$\begin{aligned} \hat{\mathbf{L}}^2 &= -\hbar^2 \mathbf{D}_\perp^2 + S^2 \\ &= \frac{-\hbar^2}{\sin \theta} \left(\partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin \theta} (\partial_\varphi + i s (1 - \cos \theta))^2 \right) + \hbar^2 s^2. \end{aligned} \quad (231)$$

- A key property of $\hat{\mathbf{L}}^2$ is that it commutes with any component of the angular momentum operator,

$$[\hat{\mathbf{L}}^2, \hat{L}_a] = 0 \quad (\text{for } a = x, y, z), \quad (232)$$

or simply written as $[\hat{\mathbf{L}}^2, \hat{\mathbf{L}}] = 0$ in vector form. It is commonly referred to as the **Casimir operator** of the $\mathfrak{so}(3)$ algebra — an element in the Lie algebra that *commutes* with all its *generators*.

**Exc
48**

Proof Eq. (232) based on Eq. (229).

Given $[\hat{\mathbf{L}}^2, \hat{L}_z] = 0$ (i.e. $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ are commuting operators), they can be *simultaneously diagonalized* by a set of **common eigenstates**, which turns out to be the *monopole harmonics*

$Y_{slm}(\theta, \varphi)$,

$$\begin{aligned}\hat{\mathbf{L}}^2 Y_{slm}(\theta, \varphi) &= \hbar^2 l(l+1) Y_{slm}(\theta, \varphi), \\ \hat{L}_z Y_{slm}(\theta, \varphi) &= \hbar m Y_{slm}(\theta, \varphi).\end{aligned}\tag{233}$$

**Exc
49**

Verify Eq. (233).

- The quantum numbers are reinterpreted as
 - $l = s, s+1, s+2, \dots \in s+\mathbb{N}$ - **angular quantum number** (labeling quantized total angular momentum),
 - $m = -l, -l+1, \dots, l-1, l$ - **magnetic quantum number** (labeling quantized z -component of angular momentum).

■ Spin-1/2

The smallest *non-trivial* monopole strength is

$$s = 1/2,\tag{234}$$

corresponding to a quantum spinning top with *axial angular momentum*

$$S = \hbar s = \frac{\hbar}{2}.\tag{235}$$

The total angular momentum can not be smaller than S .

The *lowest* Landau level is achieved at $l = s = 1/2$, where the total angular momentum saturates its minimal value $S = \hbar/2$, realizing a spin-1/2 system. In this case:

- There are only two options for the quantum number m

$$m = \pm \frac{1}{2},\tag{236}$$

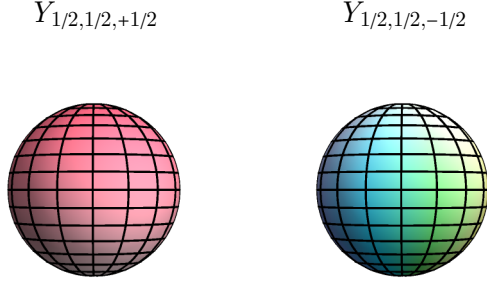
corresponding to the *up* and *down* spin states.

- The corresponding *monopole harmonics* wave functions are

$$Y_{1/2,1/2,m}(\theta, \varphi) = \frac{1}{\sqrt{4\pi}} e^{i(m-1/2)\varphi} (1 - \cos \theta)^{\frac{1/2-m}{2}} (1 + \cos \theta)^{\frac{1/2+m}{2}},\tag{237}$$

or respectively as

$$\begin{aligned}Y_{1/2,1/2,+1/2}(\theta, \varphi) &= \frac{1}{\sqrt{4\pi}} \sqrt{1 + \cos \theta}, \\ Y_{1/2,1/2,-1/2}(\theta, \varphi) &= \frac{1}{\sqrt{4\pi}} e^{-i\varphi} \sqrt{1 - \cos \theta}.\end{aligned}\tag{238}$$



- **Angular momentum** eigenvalues:

$$\begin{aligned}\hat{\mathbf{L}}^2 Y_{1/2,1/2,\pm 1/2} &= \frac{3}{4} \hbar^2 Y_{1/2,1/2,\pm 1/2}, \\ \hat{L}_z Y_{1/2,1/2,\pm 1/2} &= \pm \frac{1}{2} \hbar Y_{1/2,1/2,\pm 1/2}.\end{aligned}\tag{239}$$

- **Energy** eigenvalue (2-fold degenerated)

$$\hat{H} Y_{1/2,1/2,\pm 1/2} = \left(\frac{\hbar^2}{4 I_t} + \frac{\hbar^2}{8 I_a} \right) Y_{1/2,1/2,\pm 1/2}.\tag{240}$$

Puzzle: It seems that $Y_{1/2,1/2,-1/2}(\theta, \varphi)$ is not single-valued at the *south pole* ($\theta = \pi$), is there anything wrong?

- **Topological obstruction:** Monopole harmonics can not be *globally* single-valued in a naive coordinate sense because of the *nontrivial gauge curvature* induced by the monopole.
- However, the apparent multivaluedness is a **gauge artifact**, and can be removed by the *gauge transformation*.

Solution: define the monopole harmonics on different hemispheres with different gauge choices, and switch gauge choices at the equator by gauge transformation.

- For the **northern hemisphere** ($\theta \in [0, \pi/2]$, excluding the south pole):

$$\begin{aligned}\mathbf{A}_N(\theta, \varphi) &= \frac{\hbar}{2} \frac{\cos \theta - 1}{\sin \theta} \mathbf{e}_\varphi, \\ Y_{1/2,1/2,+1/2}^N(\theta, \varphi) &= \frac{1}{\sqrt{4\pi}} \sqrt{1 + \cos \theta}, \\ Y_{1/2,1/2,-1/2}^N(\theta, \varphi) &= \frac{1}{\sqrt{4\pi}} e^{-i\varphi} \sqrt{1 - \cos \theta}.\end{aligned}\tag{241}$$

- For the **southern hemisphere** ($\theta \in [\pi/2, \pi]$, excluding the north pole):

$$\mathbf{A}_S(\theta, \varphi) = \frac{\hbar}{2} \frac{\cos \theta + 1}{\sin \theta} \mathbf{e}_\varphi,$$

$$Y_{1/2,1/2,+1/2}^S(\theta, \varphi) = \frac{1}{\sqrt{4\pi}} e^{i\varphi} \sqrt{1 + \cos \theta},$$

$$Y_{1/2,1/2,-1/2}^S(\theta, \varphi) = \frac{1}{\sqrt{4\pi}} \sqrt{1 - \cos \theta}.$$

- On the **equator** ($\theta = \pi/2$), the two gauge choices are related by a **gauge transformation**:

$$\begin{aligned} \mathbf{A}_N(\pi/2, \varphi) &= \mathbf{A}_S(\pi/2, \varphi) + \hbar e_\varphi \partial_\varphi \chi(\varphi), \\ Y_{1/2,1/2,m}^N(\pi/2, \varphi) &= e^{i\chi(\varphi)} Y_{1/2,1/2,m}^S(\pi/2, \varphi), \end{aligned} \tag{243}$$

with $\chi(\varphi) = -\varphi$.

**Exc
50**

Verify Eq. (243).

Therefore, monopole harmonics can (only) be defined by piecing different gauge patches together with gauge transformations,

$$Y_{1/2,1/2,m}(\theta, \varphi) = \begin{cases} Y_{1/2,1/2,m}^N(\theta, \varphi) & \theta \in [0, \pi/2], \\ Y_{1/2,1/2,m}^S(\theta, \varphi) & \theta \in [\pi/2, \pi], \end{cases} \tag{244}$$

such that there is no singularity on any patch, and physical quantities are all well-behaved.