■ Part I: Matrix Mechanics

■ Representation of Operator

- *Definition* of an operator:

$$\hat{O} = \sum_{ij} |i\rangle \, O_{ij} \, \langle j|. \tag{1}$$

- *Action* of an operator on a basis state:

$$\hat{O}\,|j\rangle = \sum_{i} |i\rangle \, O_{ij}. \tag{2}$$

- *Matrix representation* of an operator

$$\hat{O} \doteq \begin{pmatrix} O_{11} & O_{12} & \cdots \\ O_{21} & O_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{3}$$

Method: make a table, fill in the matrix element.
Indexing rule: from **top** to **left**.

$$\hat{O}\,|j\rangle \to O_{ij}\,|i\rangle \Rightarrow \begin{array}{c|ccc} & \cdots & j & \cdots \\ \hline \vdots & & \downarrow & \\ i & \leftarrow & O_{ij} & \\ \vdots & & & \end{array} \tag{4}$$

> **Exc 1**
> Write down matrix representations of the following single-qubit operators $\hat{X}$, $\hat{Y}$, $\hat{Z}$:
> $\hat{X}\,|0\rangle = |1\rangle$, $\hat{X}\,|1\rangle = |0\rangle$;
> $\hat{Y}\,|0\rangle = i\,|1\rangle$, $\hat{Y}\,|1\rangle = -i\,|0\rangle$;
> $\hat{Z}\,|0\rangle = |0\rangle$, $\hat{Z}\,|1\rangle = -|1\rangle$.

Using the table method:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline & & \\ 0 & & \\ 1 & & \end{array} \overset{\hat{X}|0\rangle=|1\rangle}{\Rightarrow} \begin{array}{c|cc} & 0 & 1 \\ \hline & \downarrow & \\ 0 & \downarrow & \\ 1 & \leftarrow & 1 \end{array} \overset{\hat{X}|1\rangle=|0\rangle}{\Rightarrow} \begin{array}{c|ccc} & & 0 & 1 \\ \hline & & & \downarrow \\ 0 & \leftarrow & \leftarrow & 1 \\ 1 & & 1 & \end{array} \tag{5}$$

$$\text{therefore}: \hat{X} \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$\begin{array}{c|cc} & 0 & 1 \\ \hline & & \\ 0 & & \\ 1 & & \end{array} \overset{\hat{Y}|0\rangle=i|1\rangle}{\Rightarrow} \begin{array}{c|cc} & 0 & 1 \\ \hline & \downarrow & \\ 0 & \downarrow & \\ 1 & \leftarrow & i \end{array} \overset{\hat{Y}|1\rangle=-i|0\rangle}{\Rightarrow} \begin{array}{c|ccc} & & 0 & 1 \\ \hline & & & \downarrow \\ 0 & \leftarrow & \leftarrow & -i \\ 1 & & i & \end{array} \tag{6}$$

therefore : $\hat{Y} \simeq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

$$\begin{array}{c|cc} & 0 & 1 \\ \hline & & \\ 0 & & \\ 1 & & \end{array} \overset{\hat{Z}\,|0\rangle=|0\rangle}{\Rightarrow} \begin{array}{c|cc} & 0 & 1 \\ \hline & \downarrow & \\ 0 & \leftarrow 1 & \\ 1 & & \end{array} \overset{\hat{Z}\,|1\rangle=-|1\rangle}{\Rightarrow} \begin{array}{c|cc} & 0 & 1 \\ \hline & & \downarrow \\ 0 & 1 & \downarrow \\ 1 & \leftarrow \leftarrow & -1 \end{array} \tag{7}$$

therefore : $\hat{Z} \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Exc 2 Consider a system of two qubits, called A and B. Write down the matrix representation of $\hat{X}_A$, $\hat{Z}_B$, and $\hat{X}_A\,\hat{Z}_B$ in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis.

- Rules for $\hat{X}_A$:

$$\hat{X}_A\,|0?\rangle = |1?\rangle,\ \hat{X}_A\,|1?\rangle = |0?\rangle. \tag{8}$$

  - Method I: Filling the table

$$\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & & & 1 & \\ 01 & & & & 1 \\ 10 & 1 & & & \\ 11 & & 1 & & \end{array} \Rightarrow \hat{X}_A \simeq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{9}$$

  - Method II: Tensor product

$$\begin{aligned} \underset{(4\times4)}{\hat{X}_A} &= \underset{(2\times2)}{\hat{X}} \otimes \underset{(2\times2)}{\mathbb{1}} \\ &\simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{10}$$

- Rules for $\hat{Z}_B$:

$$\hat{Z}_B\,|?0\rangle = |?0\rangle,\ \hat{Z}_B\,|?1\rangle = -|?1\rangle. \tag{11}$$

  - Method I: Filling the table

$$\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & & & \\ 01 & & -1 & & \\ 10 & & & 1 & \\ 11 & & & & -1 \end{array} \Rightarrow \hat{Z}_B \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Method II: Tensor product

$$\begin{aligned} \underset{(4\times4)}{\hat{Z}_B} &= \underset{(2\times2)}{\mathbb{1}} \otimes \underset{(2\times2)}{\hat{Z}} \\ &\doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \tag{13}$$

- Rules for $\hat{X}_A \hat{Z}_B$:

$$\begin{aligned} \hat{X}_A \hat{Z}_B |00\rangle &= |10\rangle, \\ \hat{X}_A \hat{Z}_B |01\rangle &= -|11\rangle, \\ \hat{X}_A \hat{Z}_B |10\rangle &= |00\rangle, \\ \hat{X}_A \hat{Z}_B |11\rangle &= -|01\rangle. \end{aligned} \tag{14}$$

- Method I: Filling the table

$$\begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & & & 1 & \\ 01 & & & & -1 \\ 10 & 1 & & & \\ 11 & & -1 & & \end{array} \Rightarrow \hat{X}_A \hat{Z}_B \doteq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{15}$$

- Method II: Tensor product

$$\begin{aligned} \underset{(4\times4)}{\hat{X}_A \hat{Z}_B} &= \underset{(2\times2)}{\hat{X}} \otimes \underset{(2\times2)}{\hat{Z}} \\ &\doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

- Method III: Matrix multiplication

$$\hat{X}_A \simeq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \hat{Z}_B \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{17}$$

therefore,

$$\begin{aligned} \hat{X}_A \hat{Z}_B &\simeq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{18}$$

Time Evolution

Given a Hamiltonian $\hat{H}$

$$\hat{H} = \sum_n |n\rangle E_n \langle n|, \tag{19}$$

the time-evolution unitary is given by

$$\hat{U} = e^{-i \hat{H} t} = \sum_n |n\rangle e^{-i E_n t} \langle n|. \tag{20}$$

- **State** evolution (Schrödinger picture)

$$\begin{array}{ccc} |\psi\rangle & \to & \hat{U} |\psi\rangle \\ \| & & \| \\ \sum_n \psi_n |n\rangle & \to & \sum_n \psi_n e^{-i E_n t} |n\rangle \end{array} \tag{21}$$

- **Operator** evolution (Heisenberg picture)

$$\begin{array}{ccc} \hat{O} & \to & \hat{U}^\dagger \hat{O} \hat{U} \\ \shortparallel & & \shortparallel \\ \sum_{mn} O_{mn} |m\rangle\langle n| & \to & \sum_{mn} O_{mn} e^{i(E_m - E_n)t} |m\rangle\langle n| \end{array} \tag{22}$$

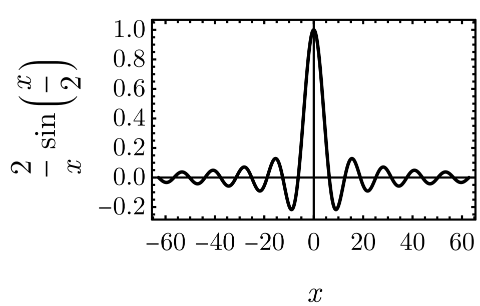
- **Expectation value** evolution (either pictures)

$$\begin{array}{ccc} \langle O\rangle_0 & \to & \langle O\rangle_t \\ \shortparallel & & \shortparallel \\ \langle\psi|\hat{O}|\psi\rangle & \to & \langle\psi|\hat{U}^\dagger \hat{O} \hat{U}|\psi\rangle \\ \shortparallel & & \shortparallel \\ \sum_{mn} \psi_m^* O_{mn} \psi_n & \to & \sum_{mn} \psi_m^* O_{mn} \psi_n e^{i(E_m - E_n)t} \end{array} \tag{23}$$

Exc 3

> Show that the **time average** of expectation value
> $\overline{\langle O\rangle} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \langle O\rangle_t$
> only depends on the diagonal matrix elements of $\hat{O}$ in the eigenbasis of $\hat{H}$.

$$\begin{aligned} \overline{\langle O\rangle} &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \langle O\rangle_t \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \sum_{mn} \psi_m^* O_{mn} \psi_n e^{i(E_m - E_n)t} \\ &= \sum_{mn} \psi_m^* O_{mn} \psi_n \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt\, e^{i(E_m - E_n)t} \\ &= \sum_{mn} \psi_m^* O_{mn} \psi_n \lim_{T\to\infty} \frac{2}{(E_m - E_n)T} \sin\left(\frac{(E_m - E_n)T}{2}\right), \end{aligned} \tag{24}$$



$$\begin{aligned} \overline{\langle O\rangle} &= \sum_{mn} \psi_m^* O_{mn} \psi_n \delta_{mn} \\ &= \sum_n \psi_n^* O_{nn} \psi_n \\ &= \sum_n O_{nn} p_n, \end{aligned} \tag{25}$$

where O_{nn} is the *diagonal* matrix element and $p_n = |\psi_n|^2$ is the **probability** for the system to be in the state $|n\rangle$. Key: *off-diagonal* matrix elements do not contribute.

▪ Measurement

Given a Hermitian observable $\hat{O}$

$$\hat{O} = \sum_k |O_k\rangle\, O_k\, \langle O_k|. \tag{26}$$

Measure the observable $\hat{O}$ on the state $|\psi\rangle$:

- Possible measurement outcomes: O_k (eigenvalues)
- Probability to observe a particular outcome O_k:

  $$p(O_k \,|\, \psi) = |\langle O_k|\psi\rangle|^2. \tag{27}$$

- After observing O_k, the state collapses to:
  - If there is no degeneracy

    $$|\psi\rangle \to |O_k\rangle\,. \tag{28}$$

  - If there is degeneracy

    $$|\psi\rangle \to \alpha_m\, |O_k, m\rangle, \tag{29}$$

    where $\alpha_m \propto \langle O_k, m|\psi\rangle$ followed by normalization.

> **Exc 4** Consider a two-qubit system, initially in arbitrary state. Measure $\hat{X}_A\, \hat{X}_B$ and $\hat{Y}_A\, \hat{Y}_B$ simultaneously. Observe $X_A\, X_B = -1$ and $Y_A\, Y_B = +1$. What is the post-measurement final state?

Let $|\psi\rangle$ be the final state, it must be eigenstates of both $\hat{X}_A\, \hat{X}_B$ and $\hat{Y}_A\, \hat{Y}_B$ with eigenvalues -1 and $+1$ respectively, i.e.

$$\begin{aligned} \hat{X}_A\, \hat{X}_B\, |\psi\rangle &= -|\psi\rangle, \\ \hat{Y}_A\, \hat{Y}_B\, |\psi\rangle &= +|\psi\rangle. \end{aligned} \tag{30}$$

To solve these equations, first write down matrix representations of operators

$$\begin{aligned} \hat{X}_A\, \hat{X}_B &\doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \hat{Y}_A\, \hat{Y}_B &\doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{31}$$

Then assume

$$|\psi\rangle \doteq \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix}, \tag{32}$$

Eq. (30) implies

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix} = - \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \psi_{11} \\ \psi_{10} \\ \psi_{01} \\ \psi_{00} \end{pmatrix} = - \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix} = \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix} \tag{33}$$

$$\Rightarrow \begin{pmatrix} -\psi_{11} \\ \psi_{10} \\ \psi_{01} \\ -\psi_{00} \end{pmatrix} = \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix}.$$

This means

$$\psi_{00} = -\psi_{11}, \ \psi_{01} = 0, \ \psi_{10} = 0. \tag{34}$$

One (normalized) solutions is

$$|\psi\rangle \doteq \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{35}$$

or expressed as

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), \tag{36}$$

which is the post-measurement final state (that is consistent with all observations).

Par II: Algebraic Methods

Harmonic Oscillator

- Annihilation and creation operators

$$\begin{cases} \hat{a} = \frac{1}{\sqrt{2}} (\hat{x} + i\,\hat{p}) \\ \hat{a}^\dagger = \frac{1}{\sqrt{2}} (\hat{x} - i\,\hat{p}) \end{cases}, \quad \begin{cases} \hat{x} = \frac{1}{\sqrt{2}} \left(\hat{a} + \hat{a}^\dagger\right) \\ \hat{p} = \frac{1}{\sqrt{2}\, i} \left(\hat{a} - \hat{a}^\dagger\right) \end{cases}. \tag{37}$$

They satisfies the commutation relation

$$[\hat{x}, \hat{p}] = i\,\mathbb{1} \Leftrightarrow \left[\hat{a}, \hat{a}^\dagger\right] = \mathbb{1}. \tag{38}$$

- Number operator

$$\hat{n} = \hat{a}^\dagger\,\hat{a}. \tag{39}$$

It defines a discrete spectrum $\hat{n}\,|n\rangle = n\,|n\rangle$ for $n \in \mathbb{N}$. Such that

$$\begin{aligned} \hat{a}\,|n\rangle &= \sqrt{n}\;|n-1\rangle, \\ \hat{a}^\dagger\,|n\rangle &= \sqrt{n+1}\;|n+1\rangle. \end{aligned} \tag{40}$$

- Hamiltonian

$$\hat{H} = \frac{1}{2}\,\hbar\,\omega\left(\hat{p}^2 + \hat{x}^2\right) = \hbar\,\omega\left(\hat{n} + \frac{\mathbb{1}}{2}\right). \tag{41}$$

  - Eigen energies

$$E_n = \hbar\,\omega\left(n + \frac{1}{2}\right). \tag{42}$$

  - Every eigenstate $|n\rangle$ can be raised from the ground state by

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(\hat{a}^\dagger)^n\,|0\rangle. \tag{43}$$

Exc 5 Calculate $\langle n|\,\hat{x}\,\hat{p}\,|n\rangle$.

Always use creation/annihilation operator to evaluate expectation values on the boson number eigenstate (energy eigenstate).

$$\langle n|\,\hat{x}\,\hat{p}\,|n\rangle = \langle n|\,\frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right)\frac{1}{\sqrt{2}\;i}\left(\hat{a} - \hat{a}^\dagger\right)|n\rangle$$

$$= \frac{1}{2\,i}\,\langle n|\left(\hat{a}\,\hat{a} - \hat{a}\,\hat{a}^\dagger + \hat{a}^\dagger\,\hat{a} - \hat{a}^\dagger\,\hat{a}^\dagger\right)|n\rangle$$

$$= \frac{1}{2\,i}\,\langle n|\left(-\,\hat{a}\,\hat{a}^\dagger + \hat{a}^\dagger\,\hat{a}\right)|n\rangle$$

$$= \frac{1}{2\,i}\,\langle n|\left(-\left[\hat{a},\,\hat{a}^{\dagger}\right]\right)|n\rangle$$
$$= \frac{1}{2\,i}\,\langle n|\,(-\mathbb{1})\,|n\rangle$$
$$= \frac{1}{2\,i}\,(-1)$$
$$= \frac{i}{2}.$$

Exc 6

> Consider 2D harmonic oscillator, described by
> $\hat{H} = \frac{1}{2}\,\hat{\boldsymbol{p}}^2 + \frac{1}{2}\,\hat{\boldsymbol{x}}^2$,
> where $\hat{\boldsymbol{p}} = (\hat{p}_1,\,\hat{p}_2)$ and $\hat{\boldsymbol{x}} = (\hat{x}_1,\,\hat{x}_2)$ with $\left[\hat{x}_i,\,\hat{x}_j\right] = \left[\hat{p}_i,\,\hat{p}_j\right] = 0$ and $\left[\hat{x}_i,\,\hat{p}_j\right] = i\,\delta_{ij}\,\mathbb{1}$. Find the eigen energies of $\hat{H}$ and the corresponding degeneracies.

Introduce two set of creation/annihilation operators

$$\begin{cases} \hat{a}_i = \frac{1}{\sqrt{2}}\,(\hat{x}_i + i\,\hat{p}_i) \\ \hat{a}_i^{\dagger} = \frac{1}{\sqrt{2}}\,(\hat{x}_i - i\,\hat{p}_i) \end{cases}. \tag{45}$$

- Commutation relations

$$\left[\hat{a}_i,\,\hat{a}_j\right] = \left[\hat{a}_i^{\dagger},\,\hat{a}_j^{\dagger}\right] = 0,\;\left[\hat{a}_i,\,\hat{a}_j^{\dagger}\right] = \delta_{ij}\,\mathbb{1} \tag{46}$$

indicates that they are independent boson modes.

Hamiltonian

$$\begin{aligned} \hat{H} &= \frac{1}{2}\,\hat{\boldsymbol{p}}^2 + \frac{1}{2}\,\hat{\boldsymbol{x}}^2 \\ &= \frac{1}{2}\left(\hat{p}_1^2 + \hat{x}_1^2\right) + \frac{1}{2}\left(\hat{p}_2^2 + \hat{x}_2^2\right) \\ &= \left(\hat{n}_1 + \frac{\mathbb{1}}{2}\right) + \left(\hat{n}_2 + \frac{\mathbb{1}}{2}\right) \\ &= \hat{n}_1 + \hat{n}_2 + \mathbb{1} \end{aligned} \tag{47}$$

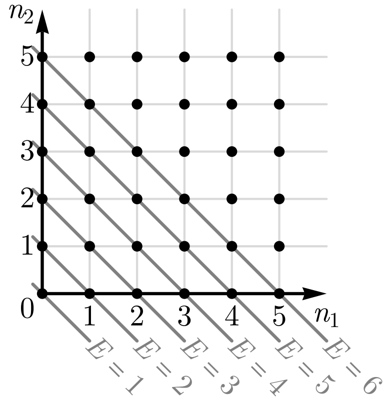
where $\hat{n}_i := \hat{a}_i^{\dagger}\,\hat{a}_i$ is the number operator for the boson mode i.

Since $\hat{n}_1$ and $\hat{n}_2$ are commuting operators, the eigenstate of $\hat{H}$ will be the joint eigenstate of $\hat{n}_1$ and $\hat{n}_2$, labeled by their corresponding quantum numbers $n_1,\,n_2 \in \mathbb{N}$:

$$\begin{aligned} \hat{n}_1\,|n_1,n_2\rangle &= n_1\,|n_1,n_2\rangle, \\ \hat{n}_2\,|n_1,n_2\rangle &= n_2\,|n_1,n_2\rangle. \end{aligned} \tag{48}$$

On these eigenstates, the eigenvalue of $\hat{H}$ will be given by

$$E = n_1 + n_2 + 1. \tag{49}$$



So the eigen energies and the corresponding degeneracies are

$$\begin{array}{cccccc} E & 1 & 2 & 3 & \cdots \\ \text{deg.} & 1 & 2 & 3 & \cdots \end{array} \tag{50}$$

Exc 7 Following (Exc 6). Define $\hat{S}_1 = \frac{1}{2}(\hat{x}_1 \hat{x}_2 + \hat{p}_1 \hat{p}_2)$, find the matrix representation of $\hat{S}_1$ in the $E = 2$ subspace

The $E = 2$ subspace is spanned by the following basis

$$\{|2,0\rangle,\ |1,1\rangle,\ |0,2\rangle\}. \tag{51}$$

To represent $\hat{S}_1$:

- Rewrite it in terms of creation/annihilation operators

$$\begin{aligned} \hat{S}_1 &= \frac{1}{2}(\hat{x}_1 \hat{x}_2 + \hat{p}_1 \hat{p}_2) \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\left(\hat{a}_1 + \hat{a}_1^\dagger\right)\frac{1}{\sqrt{2}}\left(\hat{a}_2 + \hat{a}_2^\dagger\right) + \frac{1}{\sqrt{2}\ i}\left(\hat{a}_1 - \hat{a}_1^\dagger\right)\frac{1}{\sqrt{2}\ i}\left(\hat{a}_2 - \hat{a}_2^\dagger\right)\right) \\ &= \frac{1}{4}\left(\left(\hat{a}_1 + \hat{a}_1^\dagger\right)\left(\hat{a}_2 + \hat{a}_2^\dagger\right) - \left(\hat{a}_1 - \hat{a}_1^\dagger\right)\left(\hat{a}_2 - \hat{a}_2^\dagger\right)\right) \\ &= \frac{1}{4}\left(\left(\hat{a}_1 \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger + \hat{a}_1^\dagger \hat{a}_2^\dagger\right) - \left(\hat{a}_1 \hat{a}_2 - \hat{a}_1^\dagger \hat{a}_2 - \hat{a}_1 \hat{a}_2^\dagger + \hat{a}_1^\dagger \hat{a}_2^\dagger\right)\right) \\ &= \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right). \end{aligned} \tag{52}$$

- Enumerate the action of $\hat{S}_1$ on the basis states,

$$\begin{aligned} \hat{S}_1 |2,0\rangle &= \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right)|2,0\rangle \\ &= \frac{1}{2}\left(\hat{a}_1^\dagger \hat{a}_2 |2,0\rangle + \hat{a}_2^\dagger \hat{a}_1 |2,0\rangle\right) \end{aligned}$$

$$= \frac{1}{2}\ \hat{a}_2^\dagger\ \hat{a}_1\ |2{,}0\rangle$$

$$= \frac{\sqrt{2}}{2}\ \hat{a}_2^\dagger\ |1{,}0\rangle$$

$$= \frac{1}{\sqrt{2}}\ |1{,}1\rangle,$$

$$\hat{S}_1\ |1{,}1\rangle = \frac{1}{2}\left(\hat{a}_1^\dagger\ \hat{a}_2 + \hat{a}_2^\dagger\ \hat{a}_1\right)|1{,}1\rangle$$

$$= \frac{1}{2}\left(\hat{a}_1^\dagger\ \hat{a}_2\ |1{,}1\rangle + \hat{a}_2^\dagger\ \hat{a}_1\ |1{,}1\rangle\right)$$

$$= \frac{1}{2}\left(\hat{a}_1^\dagger\ |1{,}0\rangle + \hat{a}_2^\dagger\ |0{,}1\rangle\right) \tag{54}$$

$$= \frac{\sqrt{2}}{2}\ (|2{,}0\rangle + |0{,}2\rangle)$$

$$= \frac{1}{\sqrt{2}}\ |2{,}0\rangle + \frac{1}{\sqrt{2}}\ |0{,}2\rangle$$

$$\hat{S}_1\ |0{,}2\rangle = \frac{1}{2}\left(\hat{a}_1^\dagger\ \hat{a}_2 + \hat{a}_2^\dagger\ \hat{a}_1\right)|0{,}2\rangle$$

$$= \frac{1}{2}\left(\hat{a}_1^\dagger\ \hat{a}_2\ |0{,}2\rangle + \hat{a}_2^\dagger\ \hat{a}_1\ |0{,}2\rangle\right)$$

$$= \frac{1}{2}\ \hat{a}_1^\dagger\ \hat{a}_2\ |0{,}2\rangle \tag{55}$$

$$= \frac{\sqrt{2}}{2}\ \hat{a}_1^\dagger\ |0{,}1\rangle$$

$$= \frac{1}{\sqrt{2}}\ |1{,}1\rangle.$$

- Fill the table to construct matrix representation

	$\|2{,}0\rangle$	$\|1{,}1\rangle$	$\|0{,}2\rangle$
$\|2{,}0\rangle$			
$\|1{,}1\rangle$			
$\|0{,}2\rangle$			

$\hat{S}_1\ |2{,}0\rangle = \frac{1}{\sqrt{2}}\ |1{,}1\rangle$
$\Rightarrow$

	$\|2{,}0\rangle$	$\|1{,}1\rangle$	$\|0{,}2\rangle$
$\|2{,}0\rangle$			
$\|1{,}1\rangle$	$\frac{1}{\sqrt{2}}$		
$\|0{,}2\rangle$			

$\hat{S}_1 |1,1\rangle = \frac{1}{\sqrt{2}} |2,0\rangle + \frac{1}{\sqrt{2}} |0,2\rangle$
⇒

	$\|2,0\rangle$	$\|1,1\rangle$	$\|0,2\rangle$
$\|2,0\rangle$		$\frac{1}{\sqrt{2}}$	
$\|1,1\rangle$	$\frac{1}{\sqrt{2}}$		
$\|0,2\rangle$		$\frac{1}{\sqrt{2}}$	

$\hat{S}_1 |0,2\rangle = \frac{1}{\sqrt{2}} |1,1\rangle$
⇒

	$\|2,0\rangle$	$\|1,1\rangle$	$\|0,2\rangle$
$\|2,0\rangle$		$\frac{1}{\sqrt{2}}$	
$\|1,1\rangle$	$\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$
$\|0,2\rangle$		$\frac{1}{\sqrt{2}}$	

.

Therefore, the matrix representation of $\hat{S}_1$ is

$$\hat{S}_1 \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{57}$$

■ Angular Momentum

Angular momentum operator $\hat{\boldsymbol{J}} = \left(\hat{J}_1, \hat{J}_2, \hat{J}_3\right)$ is defined by the commutation relation

$$\left[\hat{J}_a, \hat{J}_b\right] = i\, \epsilon_{abc}\, \hat{J}_c. \tag{58}$$

Based on $\hat{\boldsymbol{J}}$, we can define

- The total angular momentum operator

$$\hat{\boldsymbol{J}}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2. \tag{59}$$

- The raising and lowering operators

$$\hat{J}_\pm = \hat{J}_1 \pm i\, \hat{J}_2. \tag{60}$$

They acts on the common eigen basis $|j,m\rangle$ as

$$\begin{aligned} \hat{\boldsymbol{J}}^2 |j, m\rangle &= j(j+1)\, |j, m\rangle, \\ \hat{J}_3 |j, m\rangle &= m\, |j, m\rangle, \\ \hat{J}_\pm |j, m\rangle &= \sqrt{j(j+1) - m(m \pm 1)}\ |j, m \pm 1\rangle, \end{aligned} \tag{61}$$

where

$$\begin{aligned} j &= 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \\ m &= -j, -j+1, \ldots, j-1, j. \end{aligned} \tag{62}$$

Exc 8 Construct matrix representations of angular momentum operators $\hat{J}_1$, $\hat{J}_2$, $\hat{J}_3$ in the spin-1 subspace.

In spin-1 subspace, $j = 1$ and $m = +1, 0, -1$.

- Use $\hat{J}_3 |j, m\rangle = m |j, m\rangle$, follow the table method

	$\|1,+1\rangle$	$\|1,0\rangle$	$\|1,-1\rangle$
$\|1,+1\rangle$	$+1$		
$\|1,0\rangle$		0	
$\|1,-1\rangle$			-1

(63)

therefore,

$$\hat{J}_3 \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{64}$$

- Use $\hat{J}_+ |j, m\rangle = \sqrt{j(j+1) - m(m+1)}\, |j, m+1\rangle$, follow the table method

	$\|1,+1\rangle$	$\|1,0\rangle$	$\|1,-1\rangle$
$\|1,+1\rangle$		$\sqrt{2}$	
$\|1,0\rangle$			$\sqrt{2}$
$\|1,-1\rangle$			

(65)

therefore

$$\begin{aligned} \hat{J}_+ &\doteq \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \\ \Rightarrow \hat{J}_- = \hat{J}_+^\dagger &\doteq \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}. \end{aligned} \tag{66}$$

- The raising/lowering operators $\hat{J}_\pm$ are the keys to construct $\hat{J}_1$ and $\hat{J}_2$

$$\begin{aligned} \hat{J}_1 &= \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right) \\ &\doteq \frac{1}{2}\left(\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{67}$$

$$\hat{J}_2 = \frac{1}{2\,i}\left(\hat{J}_+ - \hat{J}_-\right)$$

$$\doteq \frac{1}{2\,i}\left(\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

In conclusion, in the spin-1 subspace

$$\hat{J}_1 \doteq \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\hat{J}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{69}$$

$$\hat{J}_3 \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$