Classical Shadow Tomography with Locally Scrambled Quantum Dynamics

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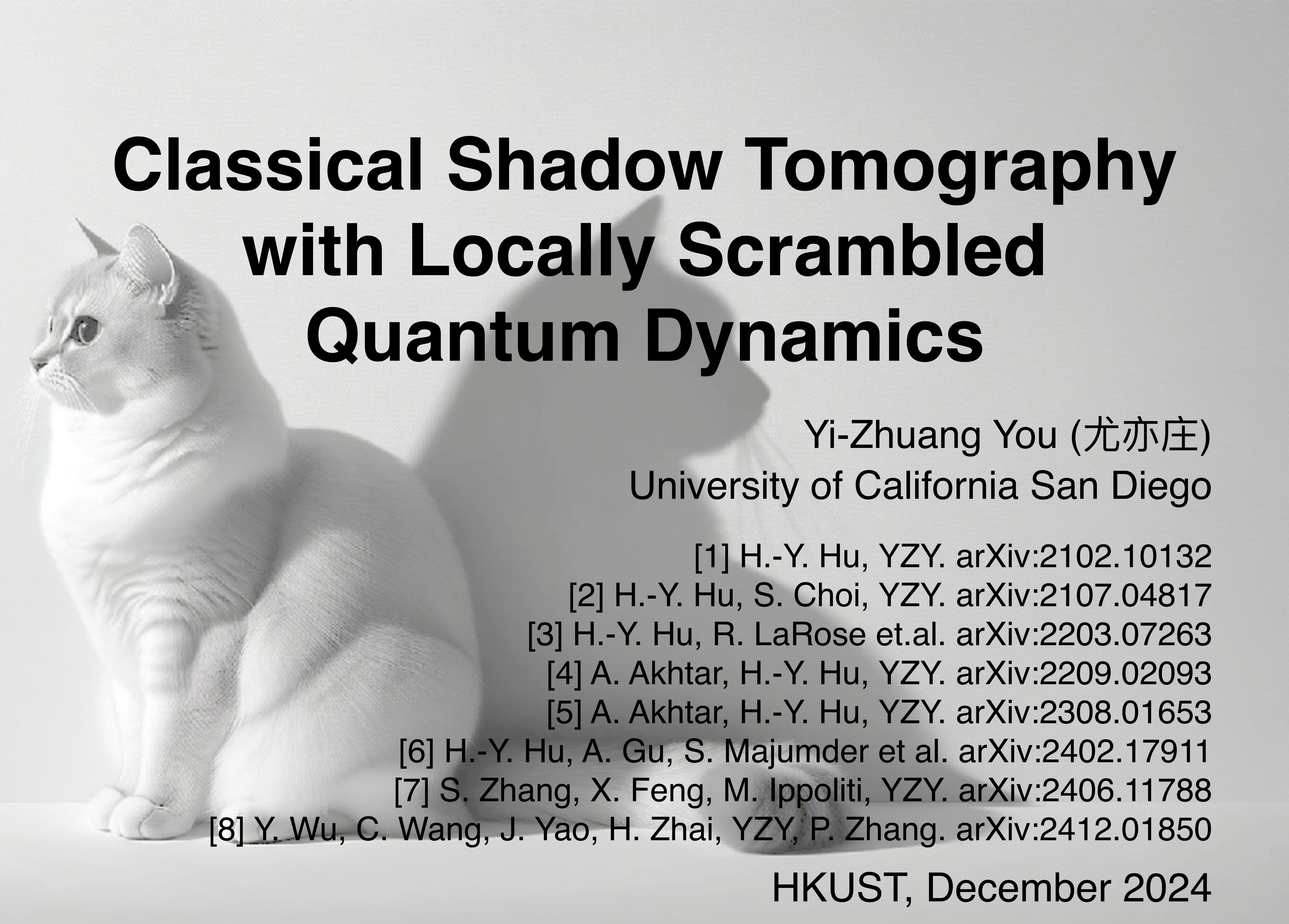
[1] H.-Y. Hu, YZY. arXiv:2102.10132
[2] H.-Y. Hu, S. Choi, YZY. arXiv:2107.04817
[3] H.-Y. Hu, R. LaRose et.al. arXiv:2203.07263
[4] A. Akhtar, H.-Y. Hu, YZY. arXiv:2209.02093
[5] A. Akhtar, H.-Y. Hu, YZY. arXiv:2308.01653
[6] H.-Y. Hu, A. Gu, S. Majumder et al. arXiv:2402.17911
[7] S. Zhang, X. Feng, M. Ippoliti, YZY. arXiv:2406.11788
[8] Y. Wu, C. Wang, J. Yao, H. Zhai, YZY, P. Zhang. arXiv:2412.01850

HKUST, December 2024

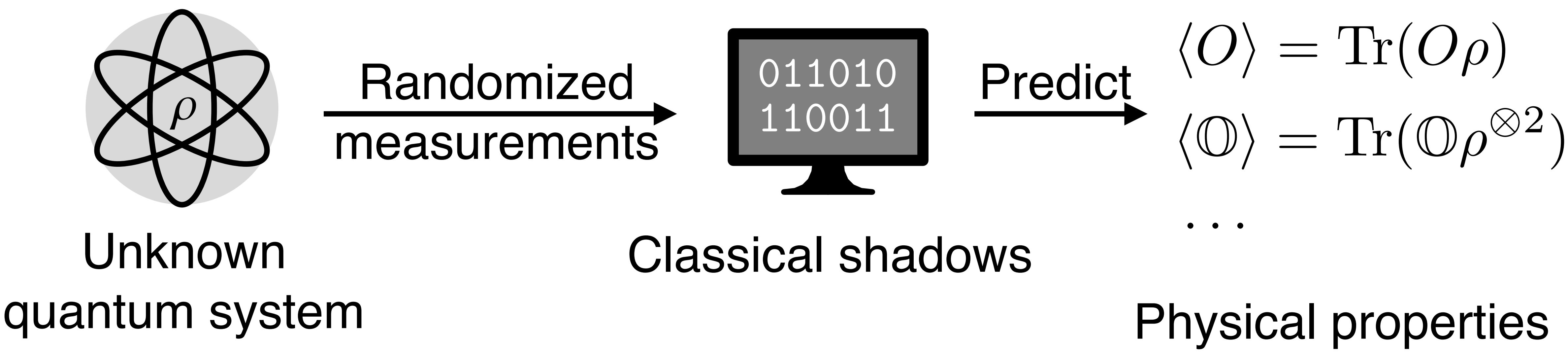


“The Shadow of a Ket” by OpenAI DALL·E 3

Quantum State Tomography

- Quantum state tomography: using repeated measurements to extract information about a quantum system.

- **Classical shadow tomography**: a general-purpose tomography scheme with superior sample efficiency



- Key idea: use randomized measurement to sample classical shadows $\hat{\sigma}$, without reconstructing ρ explicitly.

Huang, Kueng, Preskill, Nat. Phys. arXiv:2002.08953 (2020)

Elben et.al. *The randomized measurement toolbox*, Nat. Rev. Phys. (2022)

Single-Qubit Toy Example

- The density matrix of a single qubit takes the general form:

$$\rho = \frac{I + xX + yY + zZ}{2}$$

  - Pauli observables:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  - Coefficients: $(x, y, z) \in \mathbb{R}^3$ and $x^2 + y^2 + z^2 \leq 1$

- Pauli basis tomography: measure each (non-trivial) Pauli observable

$$x = \mathrm{Tr}\, \rho X = \langle X \rangle,$$
$$y = \mathrm{Tr}\, \rho Y = \langle Y \rangle,$$
$$z = \mathrm{Tr}\, \rho Z = \langle Z \rangle.$$

(Ok, but not scalable to large multi-qubit systems)

Single-Qubit Toy Example

- What happens in the measurement?

Outcome

$$\rho \xrightarrow{\text{Measure } Z} \begin{cases} +1 \xrightarrow{\phantom{\text{Collapse}}} \hat{\sigma} = \frac{1}{2}(I+Z) \\ -1 \xrightarrow{\text{Collapse}} \hat{\sigma} = \frac{1}{2}(I-Z) \end{cases}$$

  - $\hat{\sigma}$ - post-measurement state - classical snapshot.
  - The probability of observing $\hat{\sigma}$ given ρ

$$p(\hat{\sigma}|\rho) \propto \mathrm{Tr}(\rho\hat{\sigma})$$

- The idea of **classical shadow tomography**: infer an unknown state ρ by importance sampling of its classical snapshots $\hat{\sigma}$ on the quantum device via randomized measurements.

Single-Qubit Toy Example

- Propose to measure X, Y, Z with equal (1/3) probability

Suppose: $\rho = |0\rangle\langle 0| = \frac{1}{2}(I+Z)$

Possible $\hat{\sigma}$	Prior $p(\hat{\sigma})$	$\rightarrow$	Posterior $p(\hat{\sigma}\vert\rho)$
$\frac{1}{2}(I+X)$	$1/6$		$1/6$
$\frac{1}{2}(I-X)$	$1/6$		$1/6$
$\frac{1}{2}(I+Y)$	$1/6$		$1/6$
$\frac{1}{2}(I-Y)$	$1/6$		$1/6$
$\frac{1}{2}(I+Z)$	$1/6$		$1/3$
$\frac{1}{2}(I-Z)$	$1/6$		0

$$\sigma = \mathop{\mathbb{E}}_{\hat{\sigma}\in\mathcal{E}_\sigma} \hat{\sigma} = \frac{1}{2}I \qquad \sigma = \mathop{\mathbb{E}}_{\hat{\sigma}\in\mathcal{E}_{\sigma|\rho}} \hat{\sigma} = \frac{1}{2}\left(I + \frac{1}{3}Z\right)$$

Undo the depolarization by:

$$\rho = 3 \mathop{\mathbb{E}}_{\hat{\sigma}\in\mathcal{E}_{\sigma|\rho}} \hat{\sigma} - I$$

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(Depolarized from ρ)

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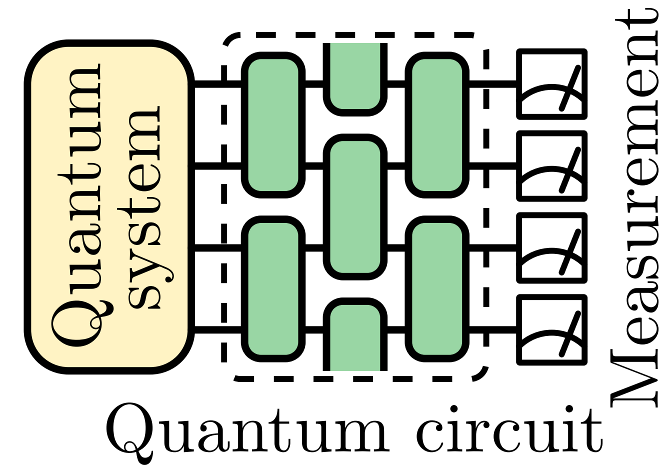
Undo the depolarization by:

$$\rho = 3 \mathop{\mathbb{E}}_{\hat{\sigma} \in \mathcal{E}_{\sigma|\rho}} \hat{\sigma} - I \longrightarrow \langle O \rangle = 3 \mathop{\mathbb{E}}_{\hat{\sigma} \in \mathcal{E}_{\sigma|\rho}} \mathrm{Tr}(\hat{\sigma} O) - \mathrm{Tr}\, O$$

Randomized Measurement

- Randomized measurement protocol
  - Prepare $\rho_{\text{full}} = \rho \otimes (|0\rangle\langle 0|)^{\otimes N_{\text{anc}}}$
  - Pick $U \sim p(U)$, perform
  $$\rho_{\text{full}} \to \rho'_{\text{full}} = U \rho_{\text{full}} U^\dagger$$
  - Measure a subset of qubits in the computational (Z) basis
  $$\rho'_{\text{full}} \to \rho_{\text{collapse}} = \frac{\Pi_{\boldsymbol{b}} \rho'_{\text{full}} \Pi_{\boldsymbol{b}}}{\text{Tr}(\Pi_{\boldsymbol{b}} \rho'_{\text{full}} \Pi_{\boldsymbol{b}})}$$
  with measurement outcome $\boldsymbol{b} \in \{0,1\}^{\times N_{\text{msr}}}$
  - Record the measurement event $(U, \boldsymbol{b})$, then repeat
- Goal: predict properties of ρ from $\{(U, \boldsymbol{b})\}$



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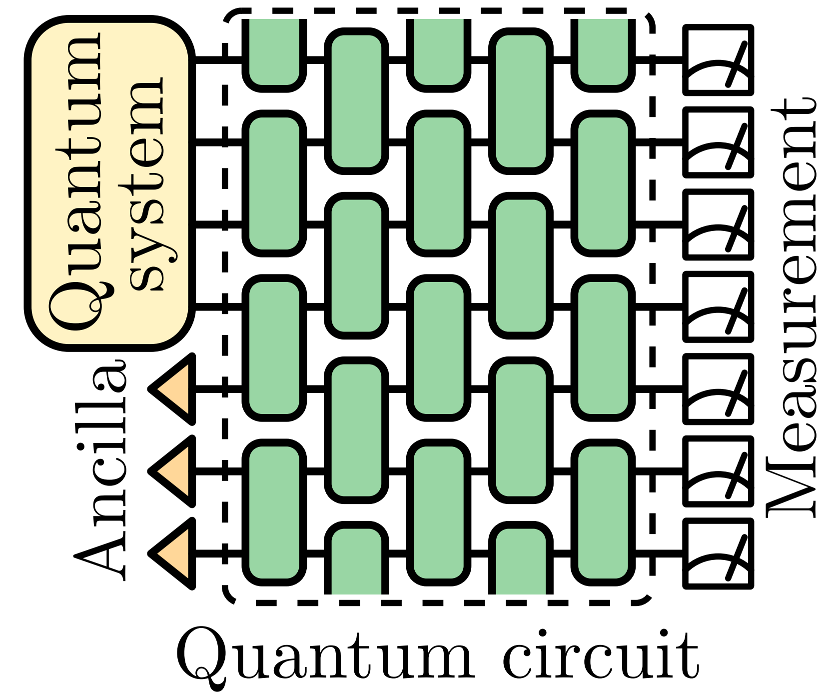

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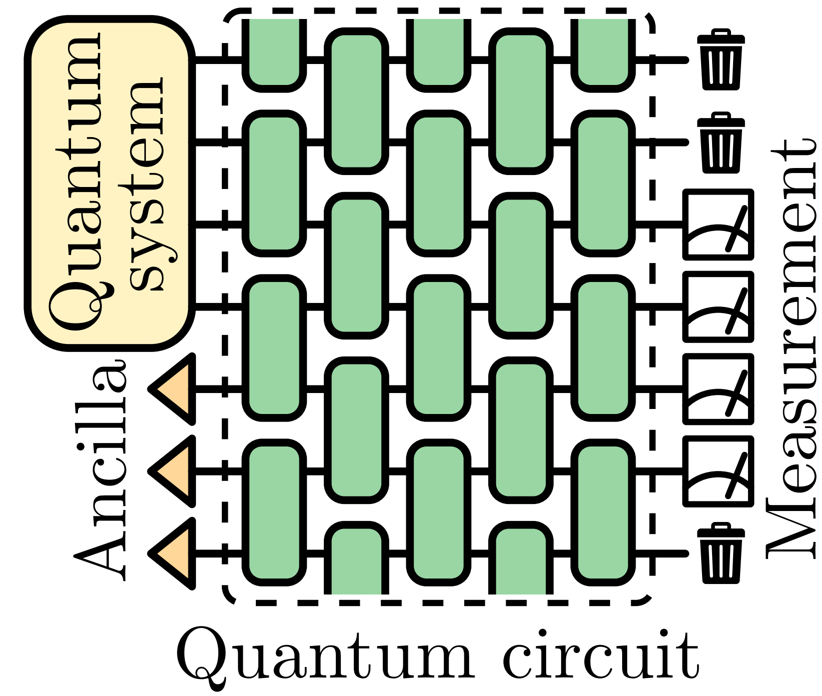
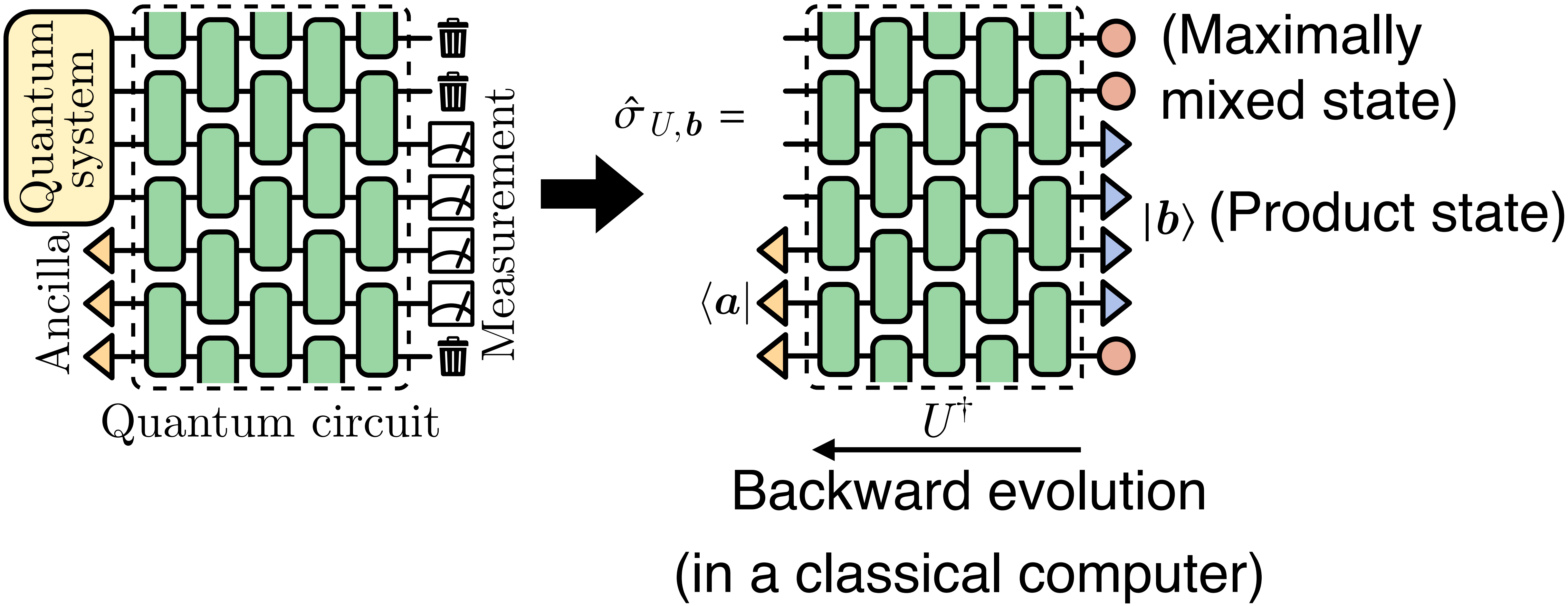


Randomized Measurement

- Each randomized measurement event $(U, \boldsymbol{b})$ is characterized by a classical shadow state $\hat{\sigma}_{U,\boldsymbol{b}}$, such that

$$p(\hat{\sigma}_{U,\boldsymbol{b}}|\rho) \propto \mathrm{Tr}(\hat{\sigma}_{U,\boldsymbol{b}}\rho)$$



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$$p(\hat{\sigma}_{U,\boldsymbol{b}}|\rho) \propto \mathrm{Tr}(\hat{\sigma}_{U,\boldsymbol{b}}\rho)$$

  - Posterior $p(\hat{\sigma}|\rho)$: the probability of observing the event $\hat{\sigma}$ given the initial state ρ.
  - Prior $p(\hat{\sigma})$: … as if there is no knowledge about ρ, i.e.

$$p(\hat{\sigma}) := p(\hat{\sigma}|\rho = \tfrac{\mathbb{1}}{\mathrm{Tr}\,\mathbb{1}})$$

- $\hat{\sigma}$ is also a quantum state (sampled by the randomized measurement), i.e. $\mathrm{Tr}\,\hat{\sigma} = 1, \hat{\sigma}^\dagger = \hat{\sigma}, \hat{\sigma} \succeq 0$

$$p(\hat{\sigma}|\rho) = (\mathrm{Tr}\,\mathbb{1})\,\mathrm{Tr}(\hat{\sigma}\rho)p(\hat{\sigma})$$

Measurement Channel

- **Measurement channel**: measure ρ prepare $\hat{\sigma}$

$$\rho \to \sigma := \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma}|\rho)} \hat{\sigma} = \mathcal{M}[\rho]$$

  - $\mathcal{M}$ is invertible, if the scheme is tomographically complete
- **Reconstruction map**: reconstruct ρ from $\hat{\sigma}$

$$\rho = \mathcal{M}^{-1}[\sigma] = \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma}|\rho)} \mathcal{M}^{-1}[\hat{\sigma}]$$

  - $\mathcal{M}^{-1}$ is *not* a physical quantum process, and can only be implemented by classical computing.
  - It specifies how to post-process the classical data to make predictions, e.g.

$$\langle O\rangle_\rho = \operatorname{Tr} O\rho = \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma}|\rho)} \operatorname{Tr}(O\mathcal{M}^{-1}[\hat{\sigma}])$$

Operator Shadow Norm

- **Sample complexity**: the number M of samples needed to control the estimation error within the level of ϵ,

$$M \sim \frac{1}{\epsilon^2} \|O\|_{\mathcal{E}_\sigma}^2$$

Huang, Kueng, Preskill (2020)

  - The scaling $\epsilon \sim 1/\sqrt{M}$ follows the large number theorem
  - The coefficient is set by the operator shadow norm

$$\|O\|_{\mathcal{E}_\sigma}^2 := \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma})} \left(\mathrm{Tr}(O\mathcal{M}^{-1}[\hat{\sigma}])\right)^2$$

$$= \frac{\mathrm{Tr}(O\mathcal{M}^{-1}[O])}{\mathrm{Tr}\,\mathbb{1}}$$

  which depends on

    - The observable O of interest
    - The randomized measurement channel $\mathcal{M}$

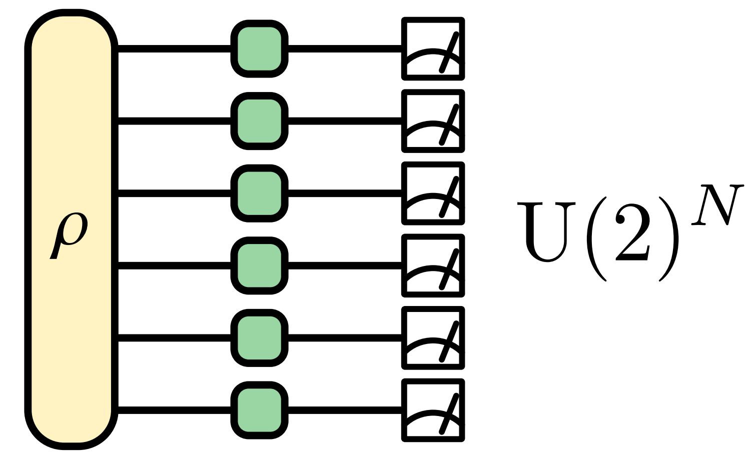
H.-Y. Hu, S. Choi, Y.-Z.You. arXiv:2107.04817

Pauli and Clifford Measurements

- Everything boils down to computing $\mathcal{M}^{-1}$.
- Know results (by 2020)
  - Randomized Pauli measurement

$$\mathcal{M}^{-1}[\sigma] = \bigotimes_i (3\sigma_i - \mathbb{1}_i)$$

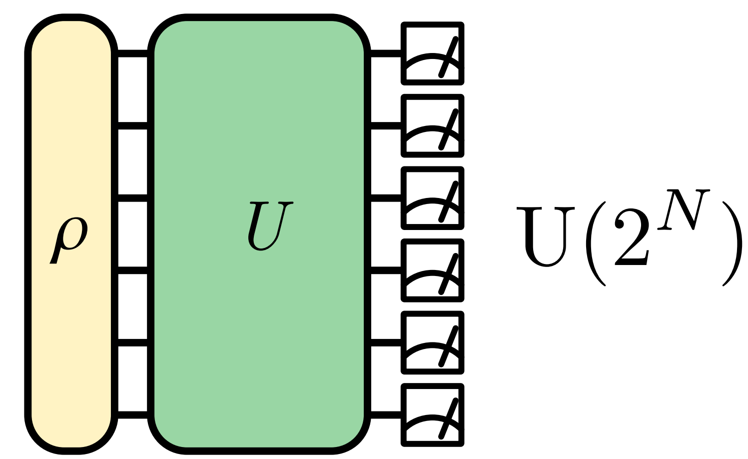
$$\|O\|^2_{\mathcal{E}_\sigma} = 3^{|\operatorname{supp} O|}$$



  - Randomized Clifford measurement

$$\mathcal{M}^{-1}[\sigma] = (2^N + 1)\sigma - \mathbb{1}$$

$$\|O\|^2_{\mathcal{E}_\sigma} \simeq \operatorname{Tr} O^2$$



- What about other more general randomized unitary ensemble (beyond Haar measure of unitary groups)?

Reconstruction Map

- By definition, $\mathcal{M}$ can be expressed as a measure-and-prepare channel

$$\mathcal{M}[\rho] = \mathop{\mathbb{E}}_{\hat{\sigma}\sim p(\hat{\sigma})} \mathrm{Tr}(\rho\hat{\sigma})\ \hat{\sigma}$$

Measure *Prepare*

  - As an operator-to-operator linear map, there are $4^N \times 4^N$ matrix elements to compute

$$\mathcal{M}[P] = \sum_{P'} \mathcal{M}_{PP'} P'$$

  - Even if $\mathcal{M}$ can be calculated, finding $\mathcal{M}^{-1}$ is challenging.
- Progress was made by considering locally-scrambled quantum dynamics.

Locally-Scrambled Shadow Tomography

- Assumption: the prior distribution of classical shadows is local basis independent — the random unitary is locally scrambled (Pauli twirled, weakly gauged).

$$\forall V = \bigotimes_i V_i \text{ with } V_i \in \mathrm{U}(2):$$

$$p(\hat{\sigma}) = p(V^\dagger \hat{\sigma} V)$$

H.-Y. Hu, S. Choi, Y.-Z.You.
arXiv:2107.04817

- $\mathcal{M}$ must be diagonal in Pauli basis (i.e. $\mathcal{M}_{PP'} \propto \delta_{PP'}$)

$$\mathcal{M}[P] = w_{\mathcal{E}_\sigma}(P)P$$

  - Eigenvalues are Pauli weights ($P \in \mathcal{P}$) of prior shadows

$$w_{\mathcal{E}_\sigma}(P) = \mathop{\mathbb{E}}_{\sigma \in \mathcal{E}_\sigma} (\mathrm{Tr}\, P\sigma)^2$$

K. Bu, D. Enshan Koh, R. J. Gracia, A. Jaffe. arXiv: 2202.03272

Locally-Scrambled Shadow Tomography

- A mild assumption: the prior distribution of classical shadows is local basis independent — the random unitary is locally scrambled (twirled).

$$\forall V = \bigotimes_i V_i \text{ with } V_i \in \mathrm{U}(2):$$

$$p(\hat{\sigma}) = p(V^\dagger \hat{\sigma} V)$$

H.-Y. Hu, S. Choi, Y.-Z.You. arXiv:2107.04817

- Inverting a diagonal matrix is straightforward

$$\mathcal{M}^{-1}[P] = \frac{1}{w_{\mathcal{E}_\sigma}(P)} P$$

- The reconstruction map is given by

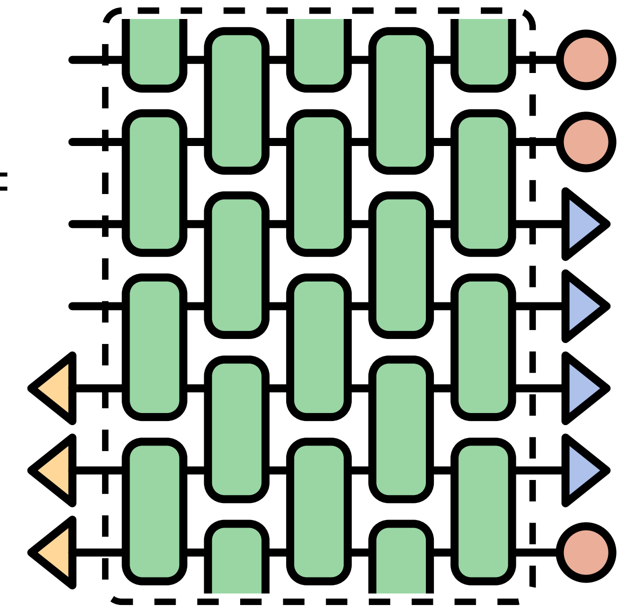
$$\boxed{\mathcal{M}^{-1}[O] = \sum_{P \in \mathcal{P}} \frac{\mathrm{Tr}\, O P}{w_{\mathcal{E}_\sigma}(P)\, \mathrm{Tr}\, \mathbb{1}} P}$$

K. Bu, D. Enshan Koh, R. J. Gracia, A. Jaffe. arXiv: 2202.03272

Locally-Scrambled Quantum Dynamics

- Classical shadows are constructed from backward quantum dynamics
  - Physical dynamics $\sigma \to \mathcal{C}[\sigma]$
  - Ensemble dynamics

$$\mathcal{E}_\sigma \to \mathcal{E}_{\mathcal{C}[\sigma]} = \{\mathcal{C}[\sigma] | \sigma \in \mathcal{E}_\sigma, \mathcal{C} \in \mathcal{E}_\mathcal{C}\}$$



  - Pauli weight dynamics

$$\boxed{w_{\mathcal{E}_{\mathcal{C}[\sigma]}}(P) = \sum_{P'} w_{\mathcal{E}_\mathcal{C}}(P, P') w_{\mathcal{E}_\sigma}(P')}$$

Definitions:

$$w_{\mathcal{E}_\sigma}(P) = \mathop{\mathbb{E}}_{\sigma \in \mathcal{E}_\sigma} (\operatorname{Tr} P\sigma)^2$$

(State Pauli weight)

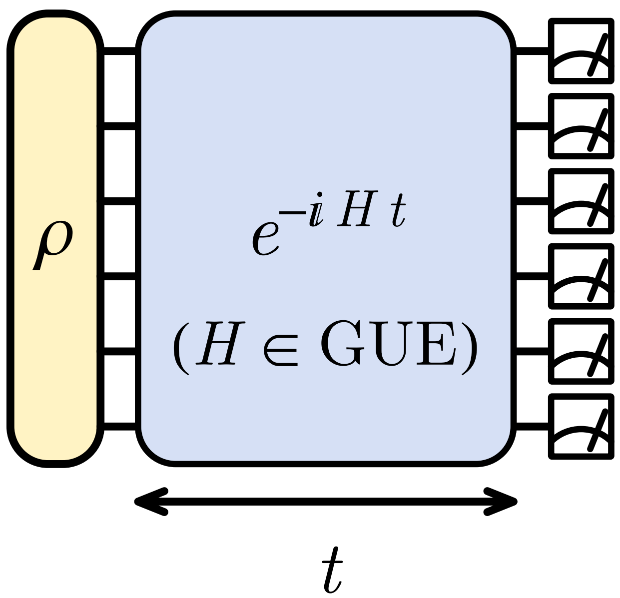
$$w_{\mathcal{E}_\mathcal{C}}(P, P') = \mathop{\mathbb{E}}_{\mathcal{C} \in \mathcal{E}_\mathcal{C}} \left(\frac{\operatorname{Tr}(P\mathcal{C}[P'])}{\operatorname{Tr} \mathbb{1}}\right)^2$$

(Channel Pauli weight)

W.-T. Kuo, A. A. Akhtar, D. P. Arovas, YZY. arXiv:1910.11351

Locally-Scrambled Tomography Schemes

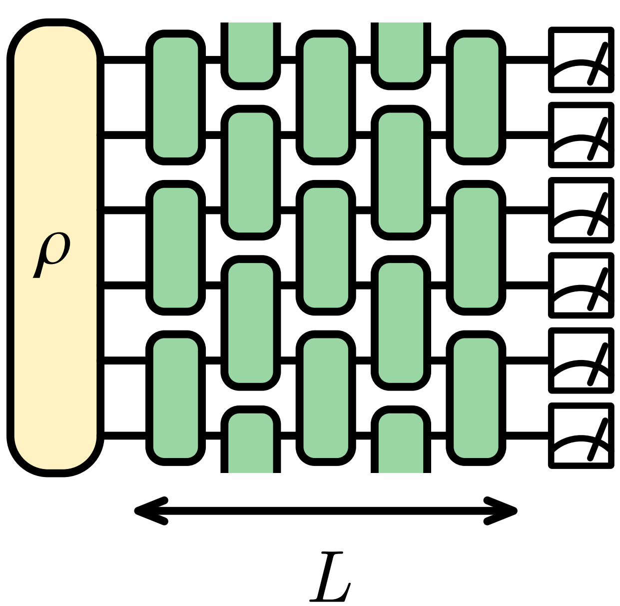
- Hamiltonian-driven shadow tomography



Hamiltonian H can be:
GUE, GOE, GSE …
SYK, ETH, Quantum simulators …

H.-Y. Hu, Y.-Z. You. arXiv:2102.10132

- Shallow shadow tomography



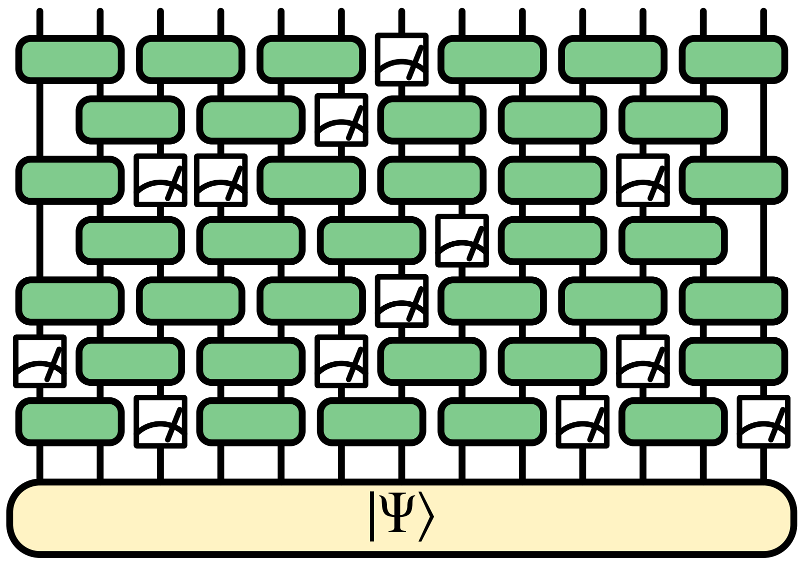
Finite-depth local random unitary
Gate: Haar, Clifford …
Friendly to NISQ devices

H.-Y. Hu, S. Choi, Y.-Z.You. arXiv:2107.04817

Locally-Scrambled Tomography Schemes

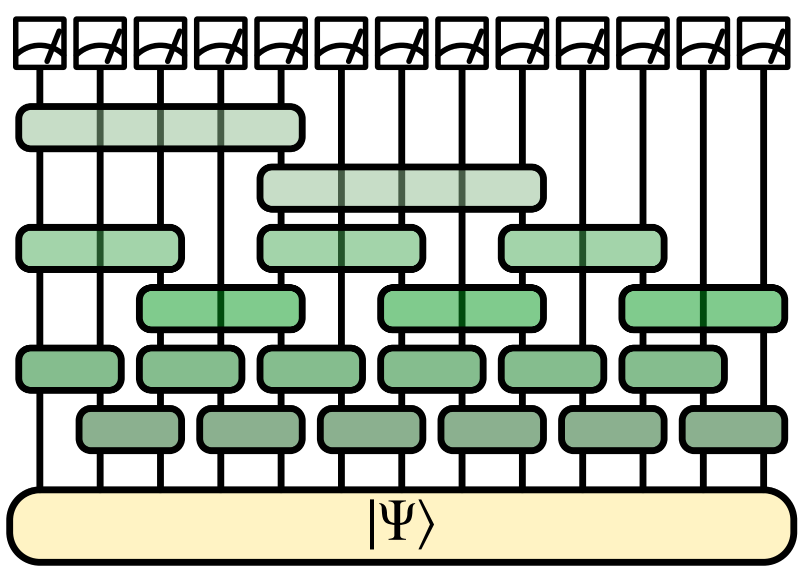
- Hybrid shadow tomography

A.Akhtar, H.-Y. Hu, YZY. arXiv:2308.01653



Intermediate measurements with a measurement rate p
— Optimal sample efficiency achieved at the measurement-induced phase transition.

- Holographic shadow tomography



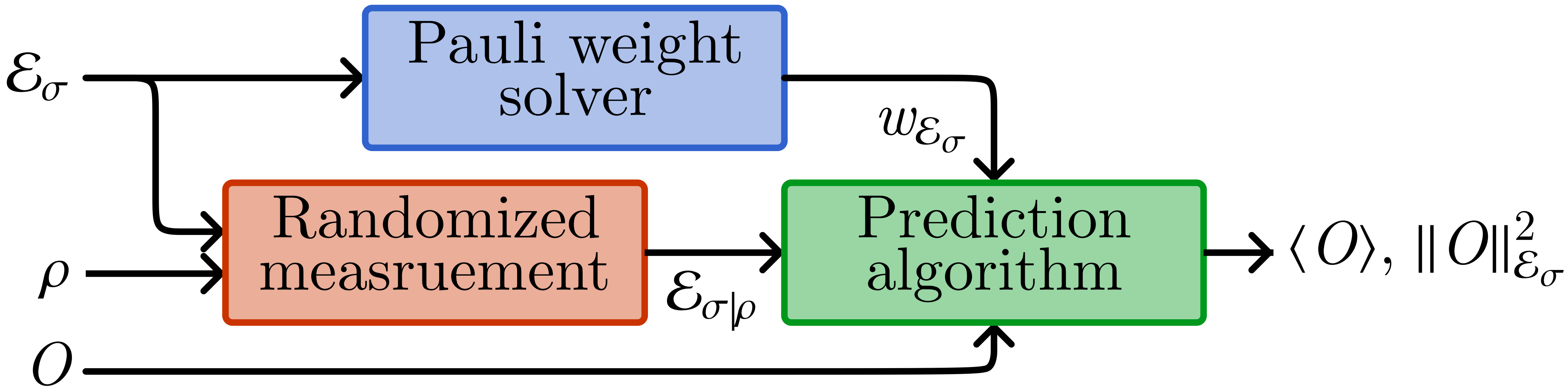
Local measurements in the holographic bulk map to scale-free measurements on the holographic boundary.

S. Zhang, X. Feng, M. Ippoliti, YZY. arXiv:2406.11788

Locally Scrambled Shadow Tomography

- Work flow

A. Akhtar, H.-Y. Hu, YZY. arXiv:2209.02093



- Snapshot constructor $(U, \boldsymbol{b}) \to \hat{\sigma}_{U,\boldsymbol{b}}$ (efficient for Clifford)
- Pauli weight solver: efficient with MPS/TN implementation
- Prediction algorithm: for $O = \sum_P o_P P$

$$\langle O \rangle = \mathop{\mathbb{E}}_{\sigma \sim p(\sigma|\rho)} \sum_P \frac{o_P \operatorname{Tr}(P\sigma)}{w_{\mathcal{E}_\sigma}(P)}$$

(Mean)

$$\|O\|^2_{\mathcal{E}_\sigma} = \sum_P \frac{|o_P|^2}{w_{\mathcal{E}_\sigma}(P)}$$

(Single-shot variance)

Scalable Tomography on Clifford Circuits

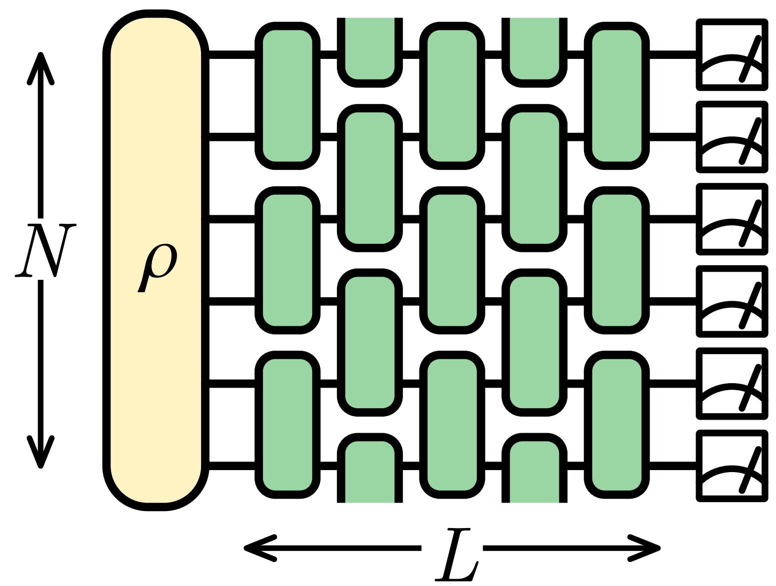
- Test states:

$$\rho_{\mathrm{GHZ}} = |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| \qquad |\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|00\cdots\rangle + |11\cdots\rangle)$$

$$\rho_{\mathrm{ZXZ}} = |\mathrm{ZXZ}\rangle\langle\mathrm{ZXZ}| \qquad Z_{i-1}X_iZ_{i+1}|\mathrm{ZXZ}\rangle = |\mathrm{ZXZ}\rangle$$

(Cluster state)

- Randomized measurement scheme:



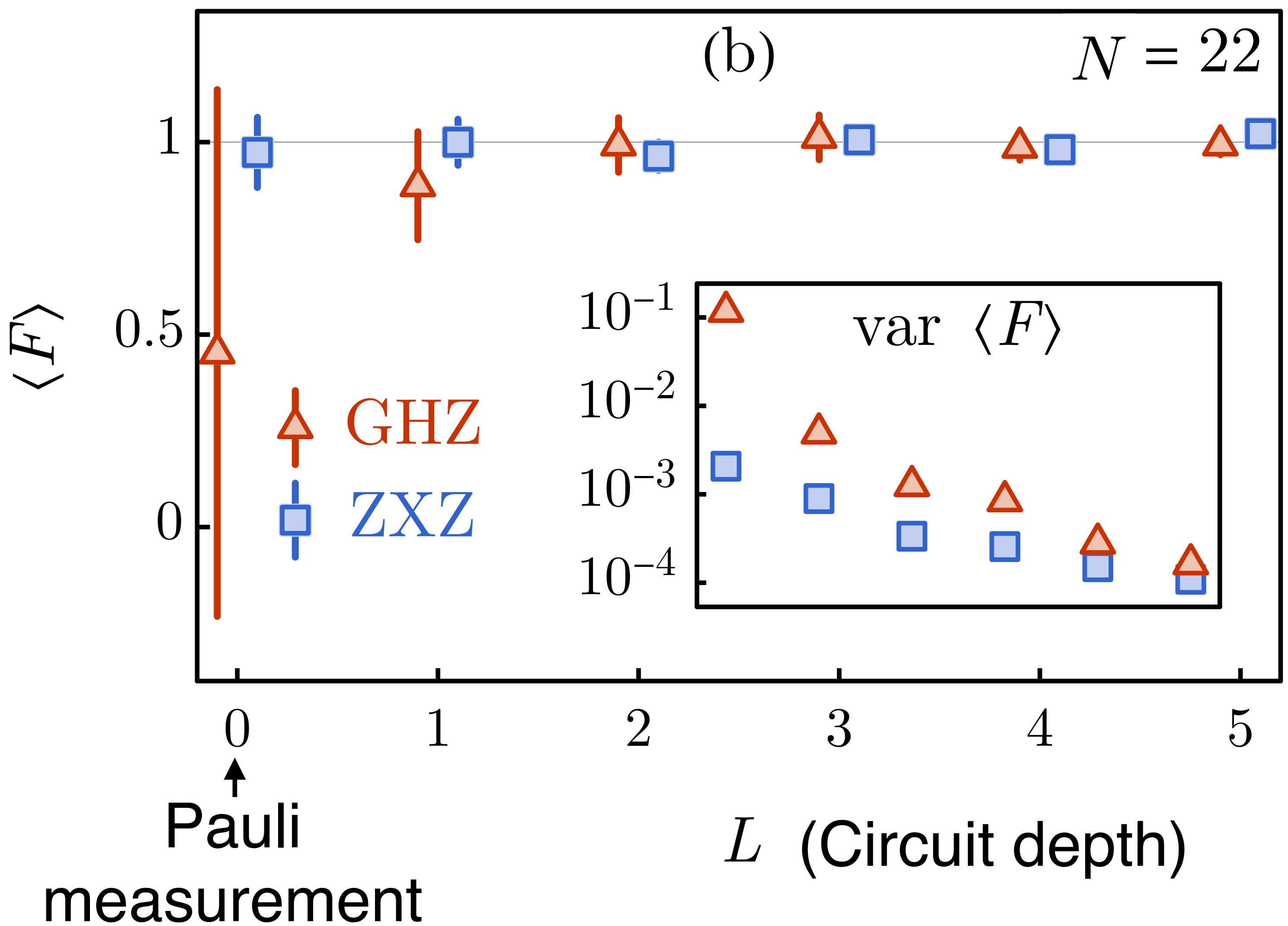
  - Brick-wall arrangement of 2-qubit random Clifford gates
  - Width (number of qubits): N
  - Depth (number of layers): L

A.A.Akhtar, H.-Y. Hu, Y.-Z.You. arXiv:2209.02093 (2022)

Fidelity Estimation

- Task 1: Estimate the fidelity (state overlap) between the reconstructed state and the original state

$$F(\rho, \rho') = \mathrm{Tr}(\rho\rho') \qquad \text{(For pure states)}$$



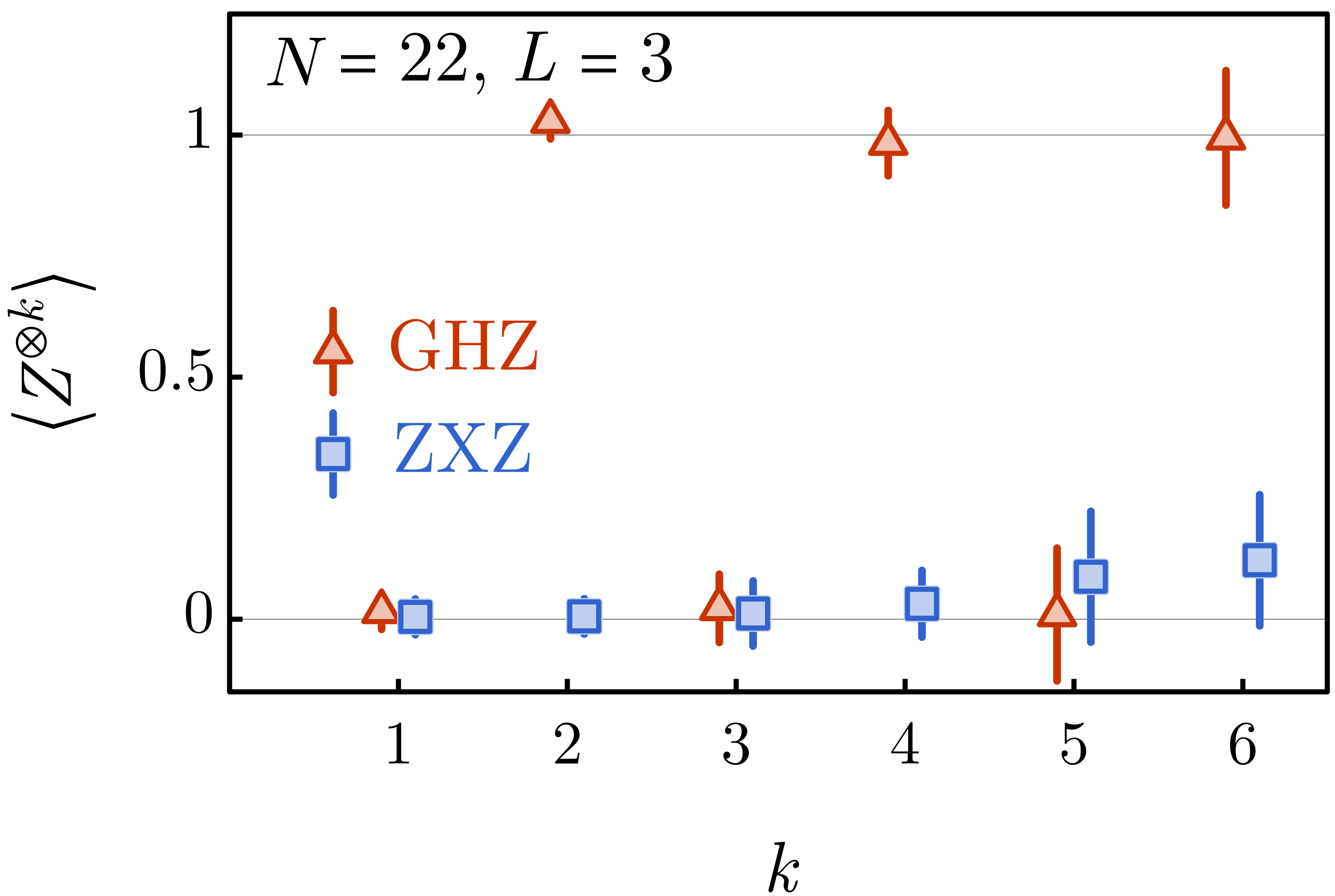
- Scalability (22 qubits)
- Reconstruction is unbiased for all circuit depth
- Variance is reduced for deeper circuits — quantum information scrambling helps!

A.A.Akhtar, H.-Y. Hu, Y.-Z.You. arXiv:2209.02093 (2022)

Pauli Operator Estimation

- Task 2: Estimate the expectation value of a Pauli string operator

$$Z^{\otimes k} := \prod_{i=1}^{k} Z_i$$



- Estimations are unbiased (converge to the ground truth)
- Variance (shadow norm) increases with the weight (size) of the Pauli string
  - Note: For Pauli measurements ($L = 0$)

$$\|Z^{\otimes k}\|^2_{\mathcal{E}_\sigma} \sim 3^k$$

A.A.Akhtar, H.-Y. Hu, Y.-Z.You. arXiv:2209.02093 (2022)

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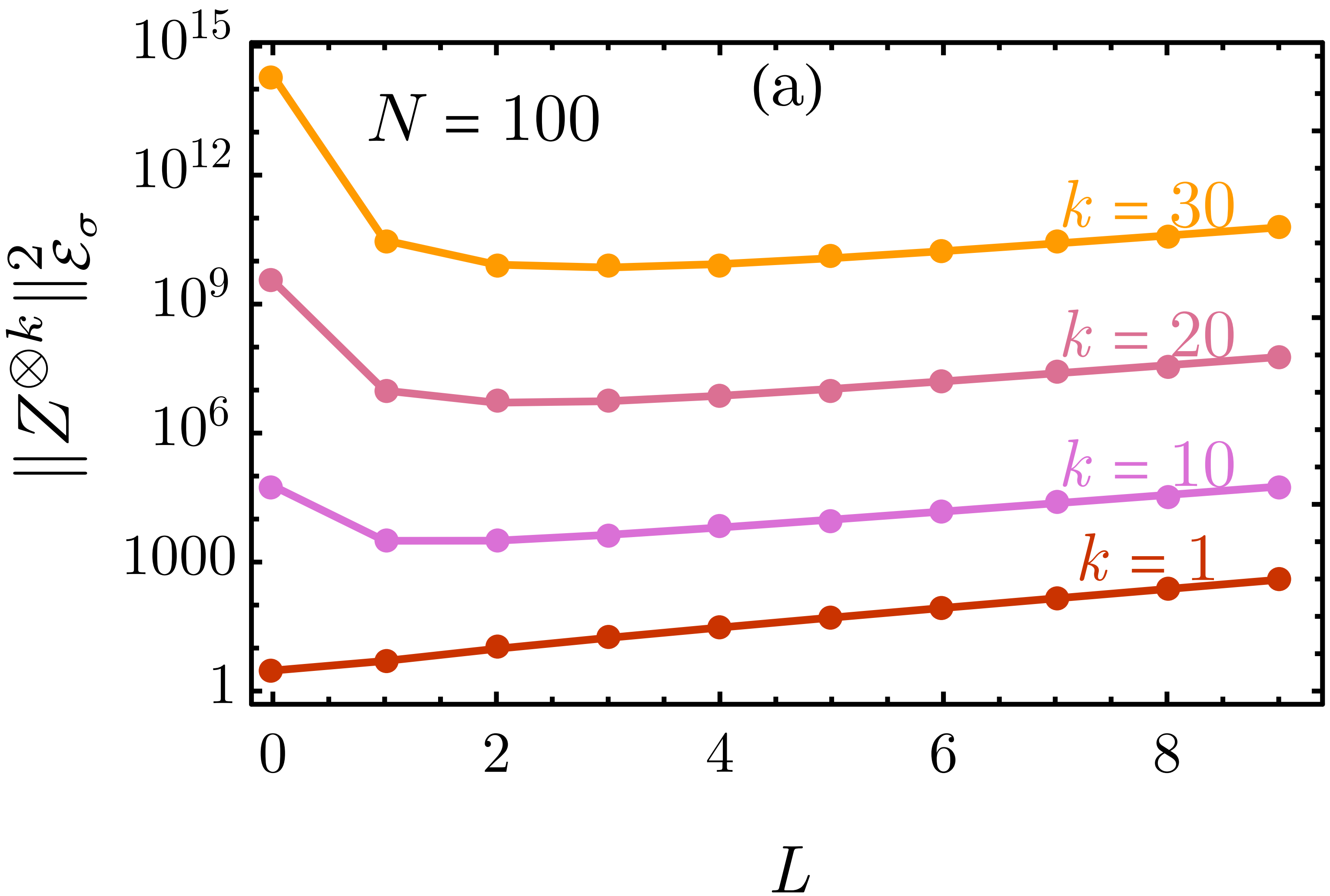
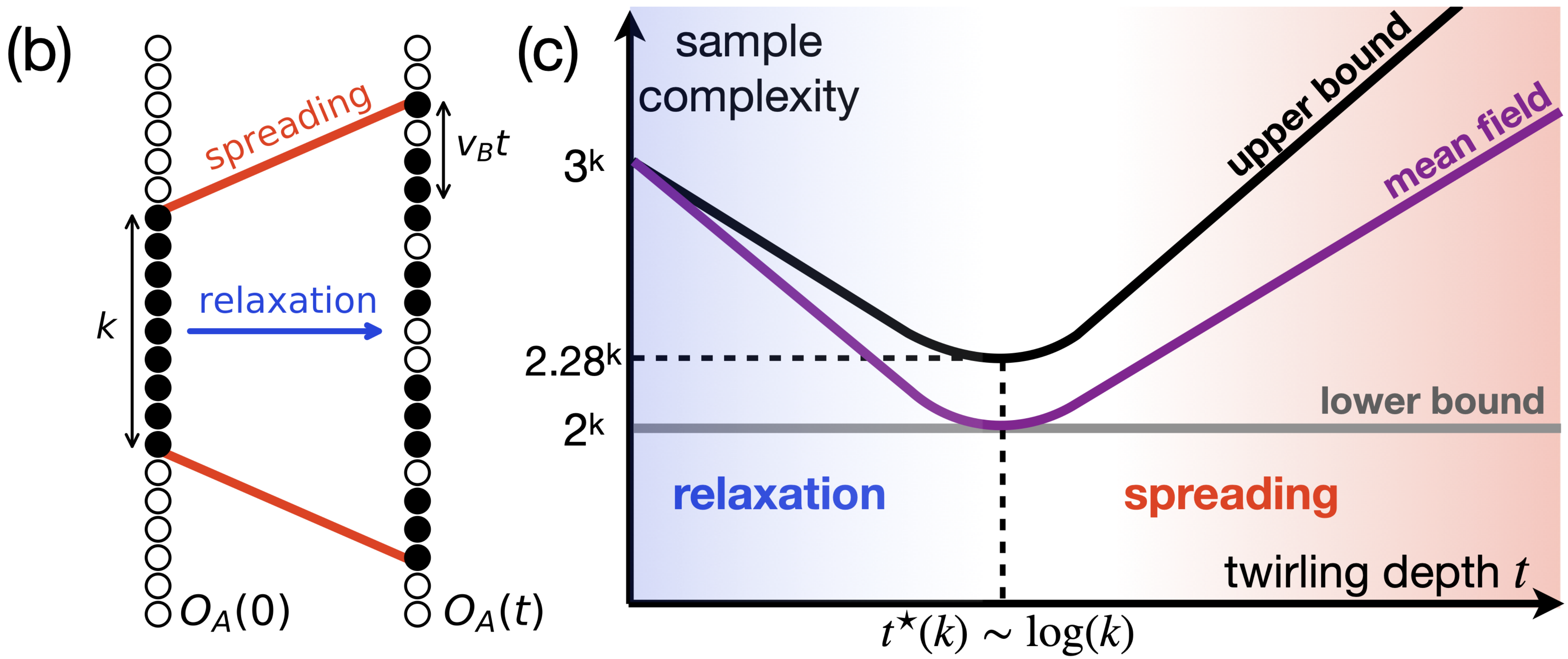


- Given the size k of the Pauli string, there is an optimal circuit depth L^* minimizing the shadow norm (sample complexity)

A.A.Akhtar, H.-Y. Hu, Y.-Z.You. arXiv:2209.02093 (2022)

Pauli Operator Estimation

- Considering continuous time limit, operator dynamics $O_A \to O_A(t) = U O_A U^\dagger$ (with $k = |A|$)



- Shadow norm ~ $3^{k_{\text{eff}}}$

$$k_{\text{eff}} \sim \left(\tfrac{3}{4} + \tfrac{1}{4}e^{-\gamma t}\right)(k + 2 v_B t) \xrightarrow{\text{Min}} t^* \sim \gamma^{-1} \log k$$

M. Ippoliti, Y. Li, T. Rakovszky, V. Khemani arXiv:2212.11963

Pauli Operator Estimation

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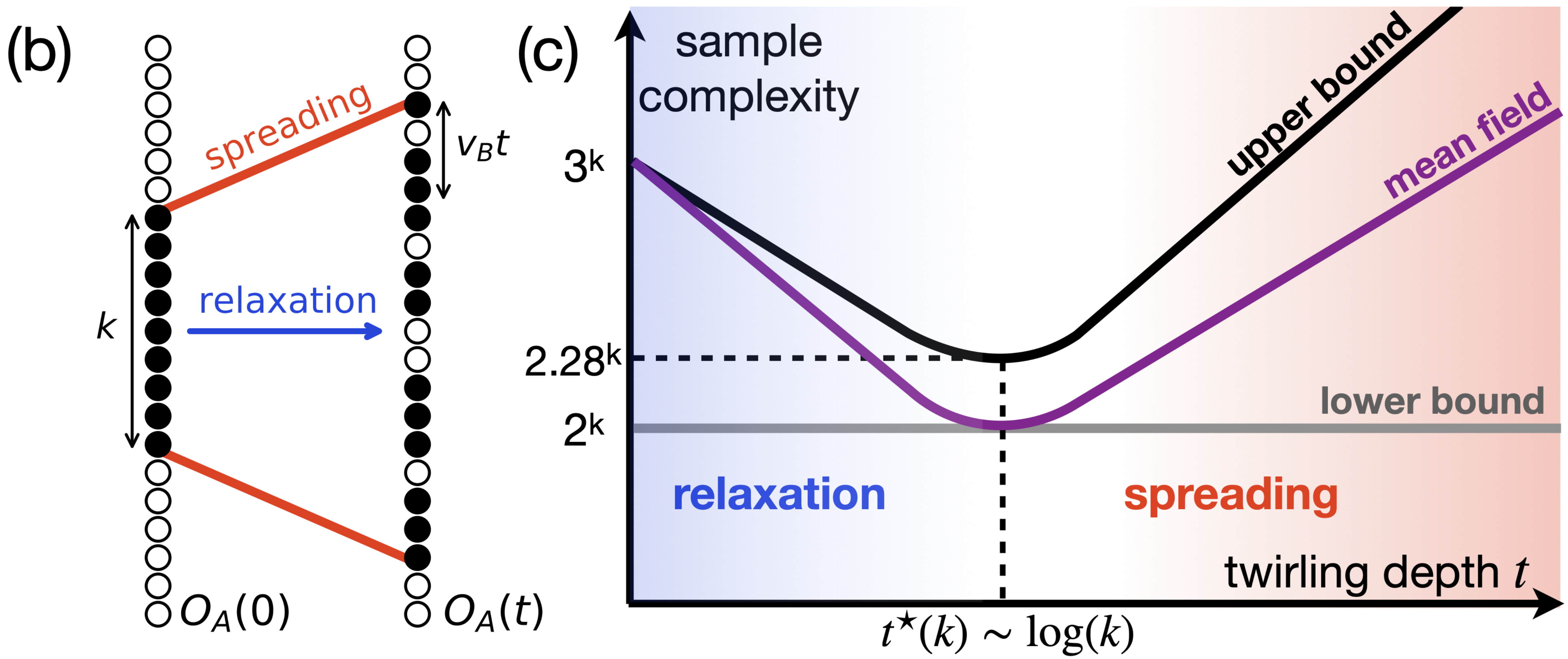


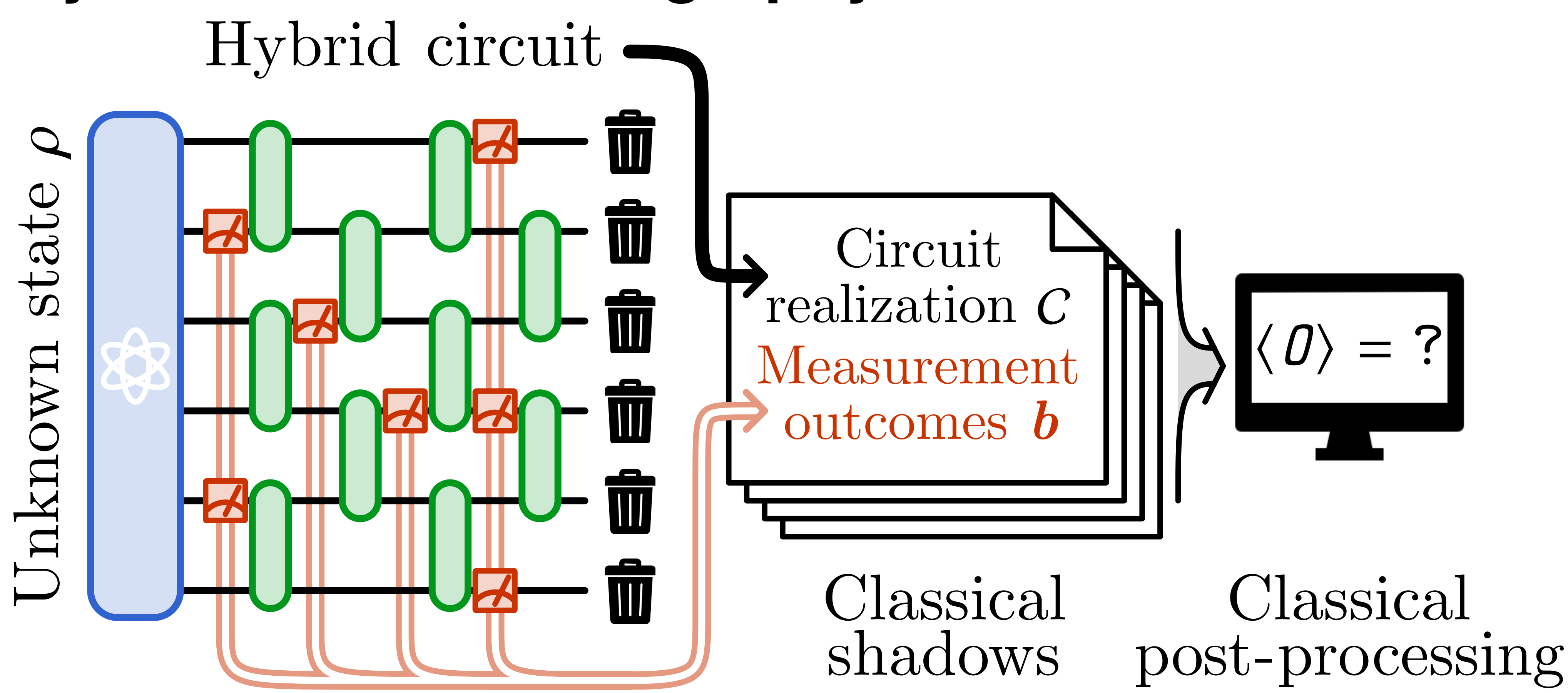
- At optimal circuit depth t^* shadow norm scales with k with a smaller base

$$2^k \lesssim \|O_A(t)\|^2_{\mathcal{E}_\sigma} \lesssim 3^{\frac{3}{4}k} \approx 2.28^k$$

M. Ippoliti, Y. Li, T. Rakovszky, V. Khemani arXiv:2212.11963

Hybrid Shadow Tomography

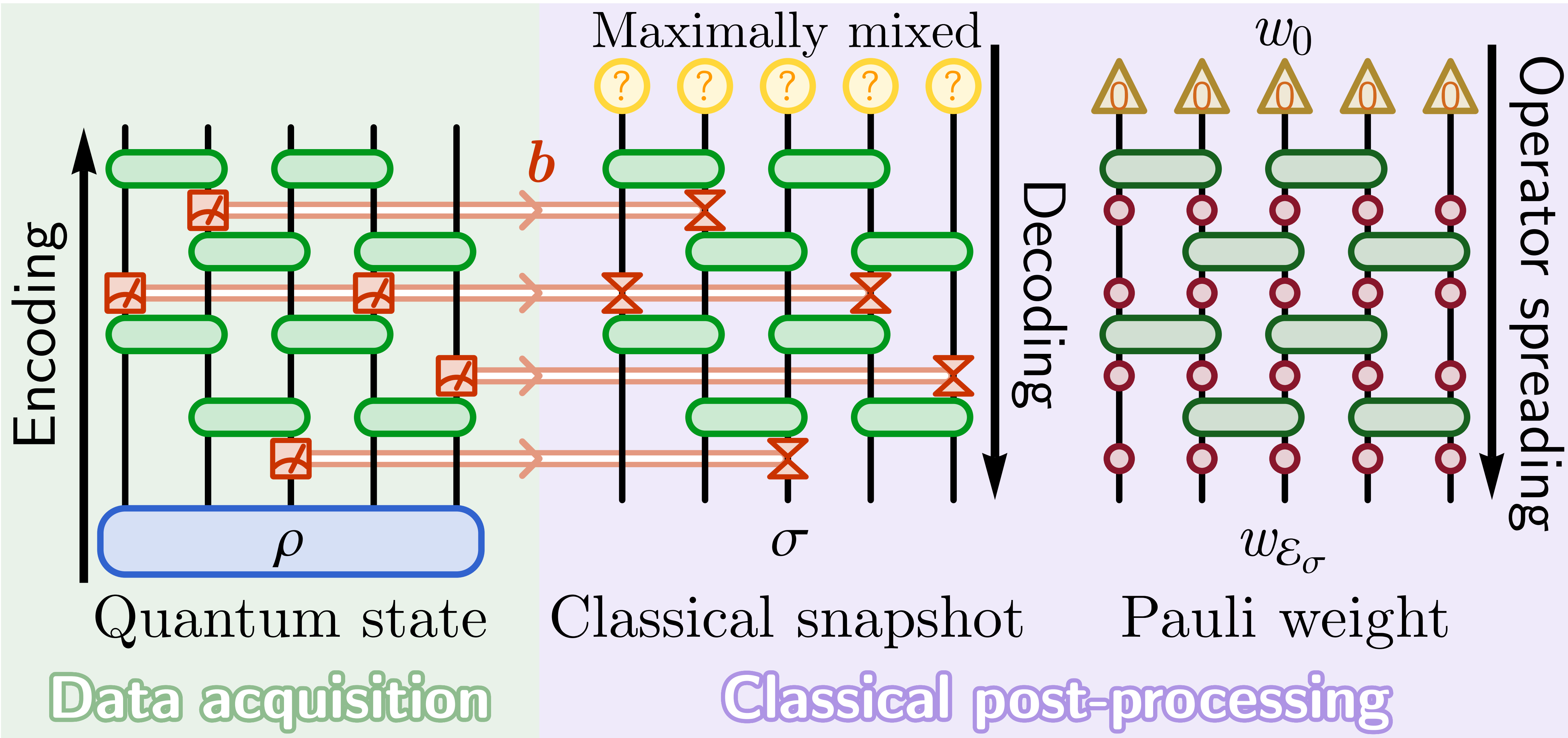
- **Shallow shadow tomography** has an advantage in sample complexity scaling, but only achievable if the circuit depth is adjusted with the size of the observable.
- Can we perform the measurement on one circuit and make predictions for observables of all sizes optimally?
  — **Hybrid shadow tomography**



A.Akhtar, H.-Y. Hu, Y.-Z. You. arXiv:2308.01653

Hybrid Shadow Tomography

- Post-processing scheme

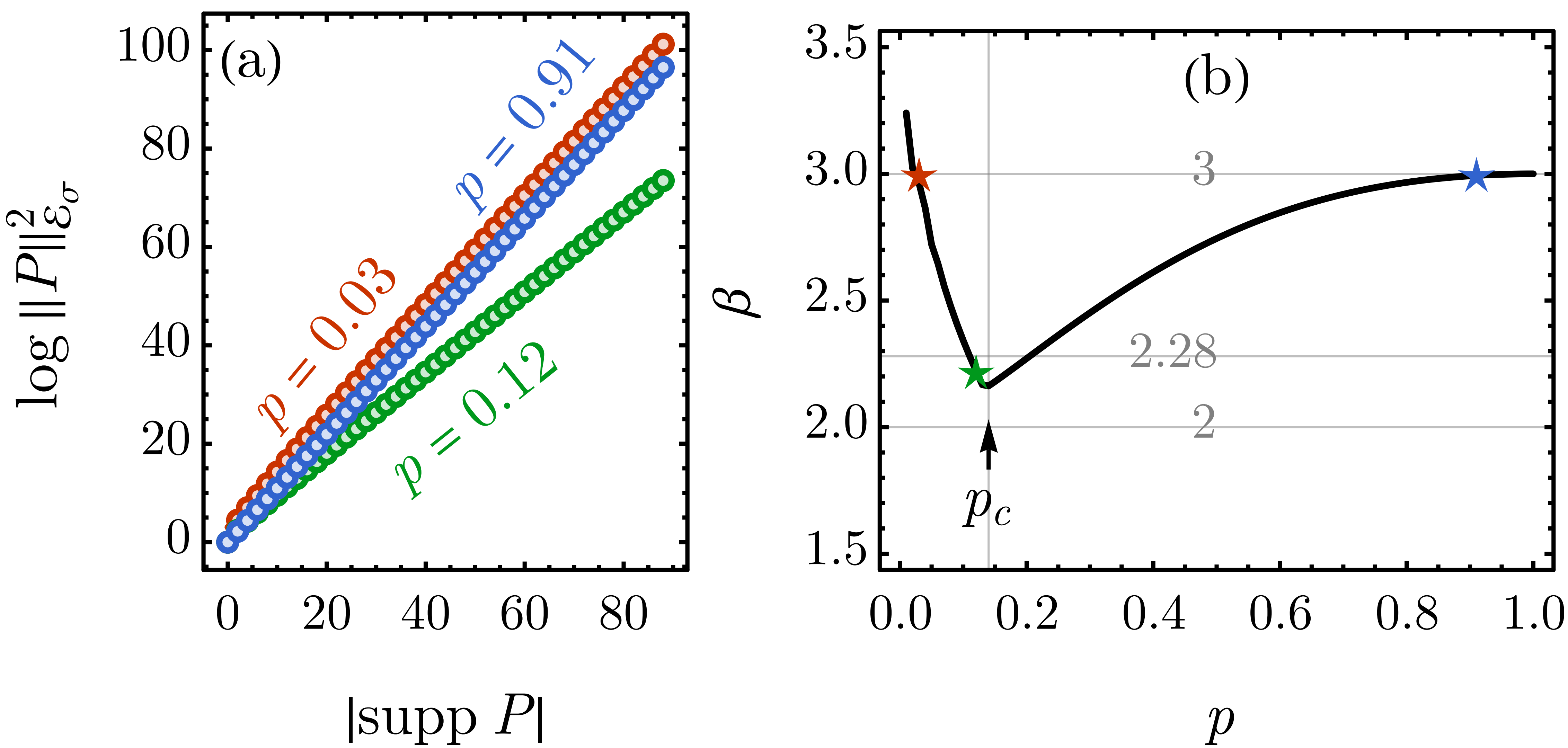


A.Akhtar, H.-Y. Hu, Y.-Z. You. arXiv:2308.01653

Hybrid Shadow Tomography

- Shadow norm scaling

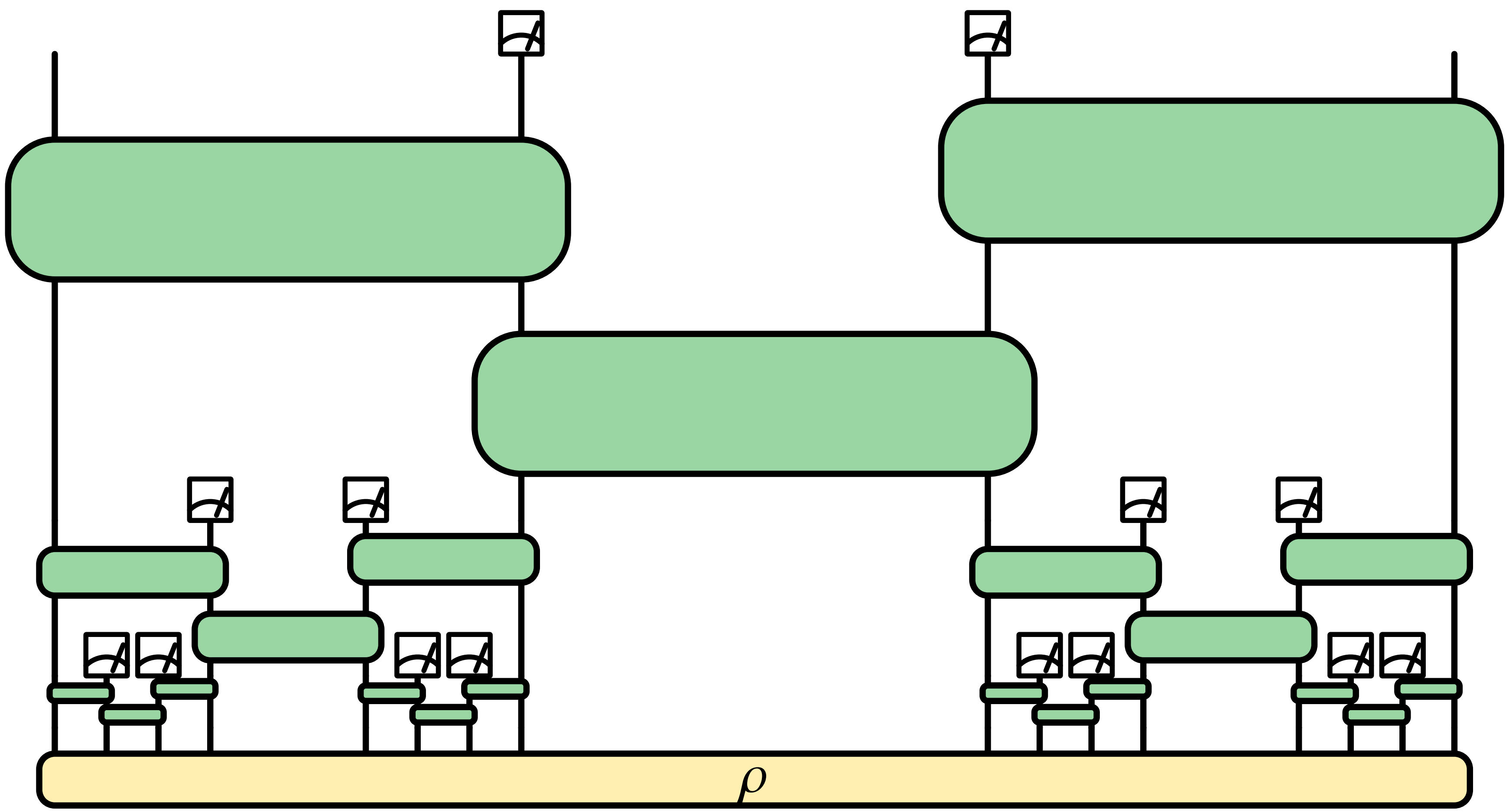
$$\|P\|^2_{\mathcal{E}_\sigma} \simeq \beta^k \mathrm{poly}(k) \qquad k = |\,\mathrm{supp}\, P|$$



A.Akhtar, H.-Y. Hu, Y.-Z. You. arXiv:2308.01653

Holographic Shadow Tomography

- However, **hybrid shadow tomography** requires fine-tuning the measurement-induced criticality.
- **Holographic shadow tomography** is automatically critical.



H.-Y. Hu, S. Choi, YZY. arXiv:2107.04817
X. Feng, S. Zhang, M. Ippoliti, YZY. arXiv:2406.11788

Holographic Shadow Tomography

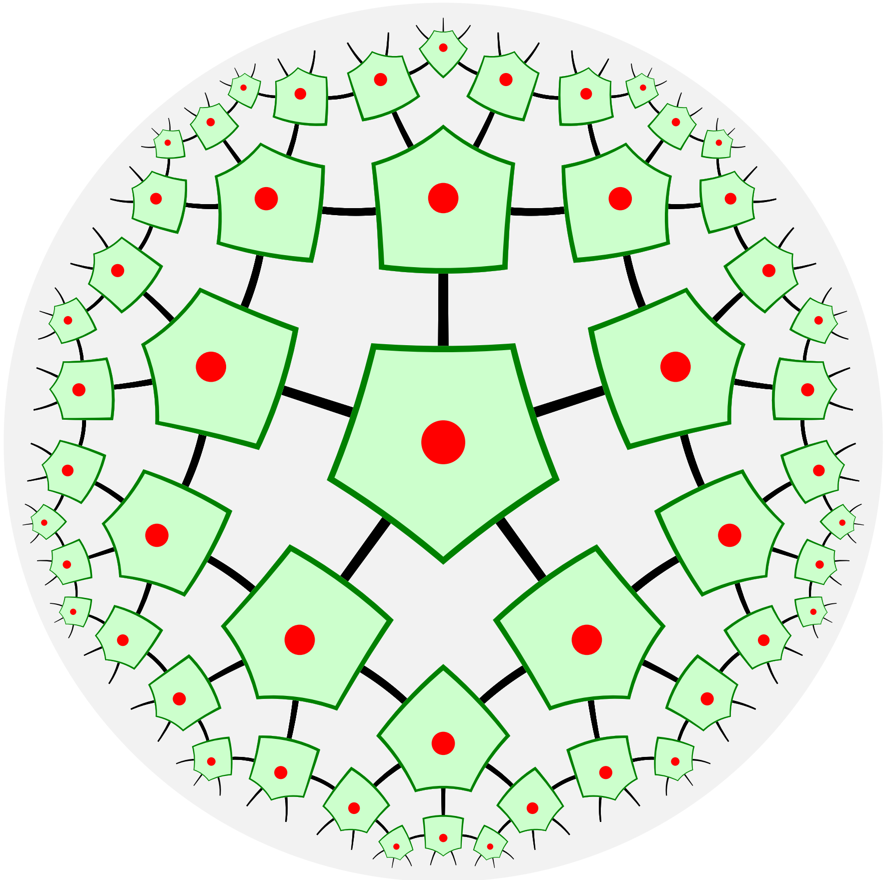
- For binary tree measurement circuit, there is a recursive approach to compute Pauli weights, from which we bound

$$\|P\|^2_{\mathcal{E}_\sigma} \leq \left(d + \frac{1}{d}\right)^k \qquad \text{(qudits of } d\text{-dim)}$$

- For holographic code circuit, in the large-d limit

$$\|P\|^2_{\mathcal{E}_\sigma} \simeq d^k k^{c_{\text{eff}} \ln d}$$

Effective central charge
in Ryu-Takayanagi formula

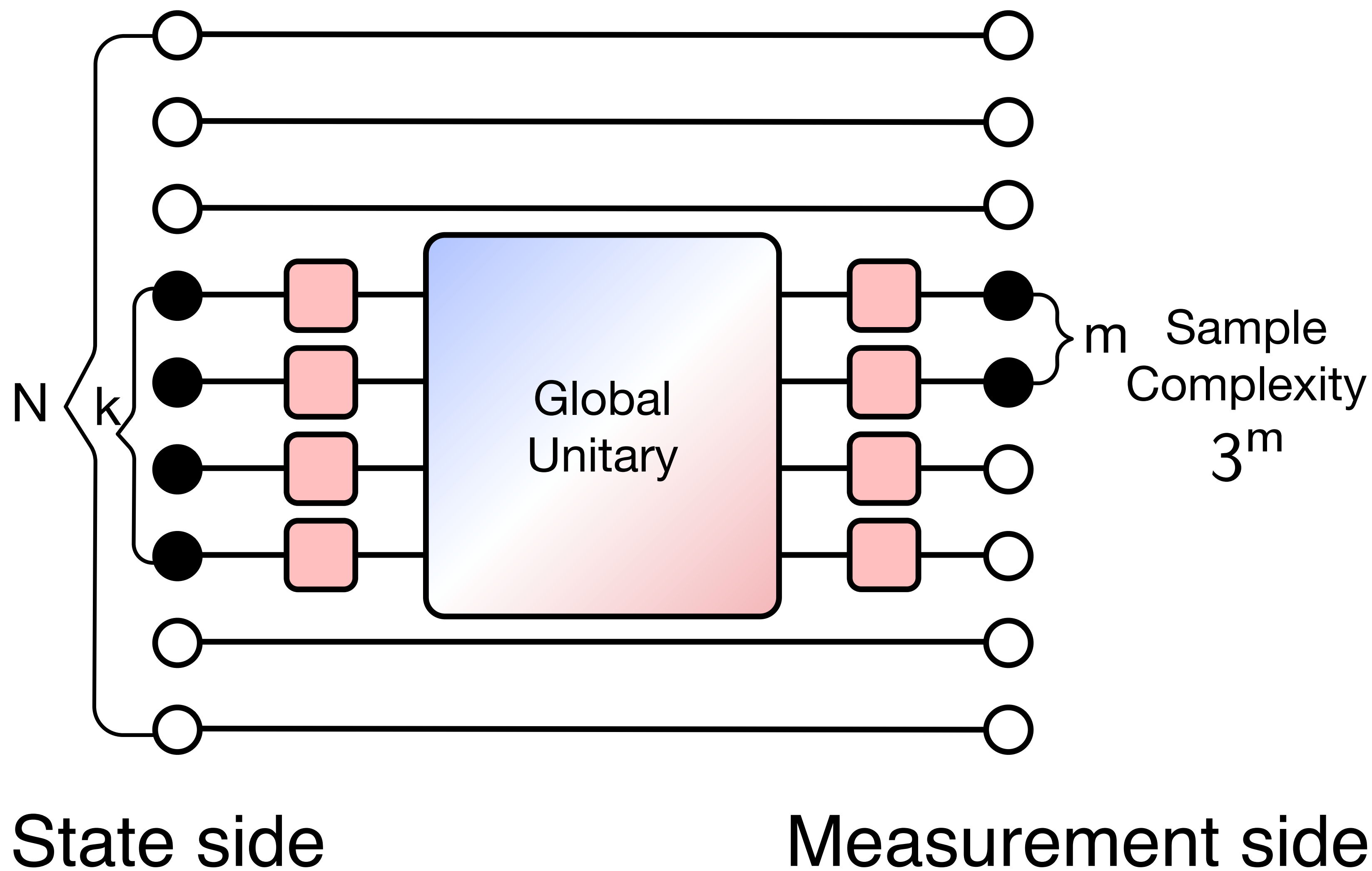


- If push to $d \to 2$, $\|P\|^2_{\mathcal{E}_\sigma} \to 2^k$

X. Feng, S. Zhang, M. Ippoliti, YZY. arXiv:2406.11788

Contractive Unitary Shadow Tomography

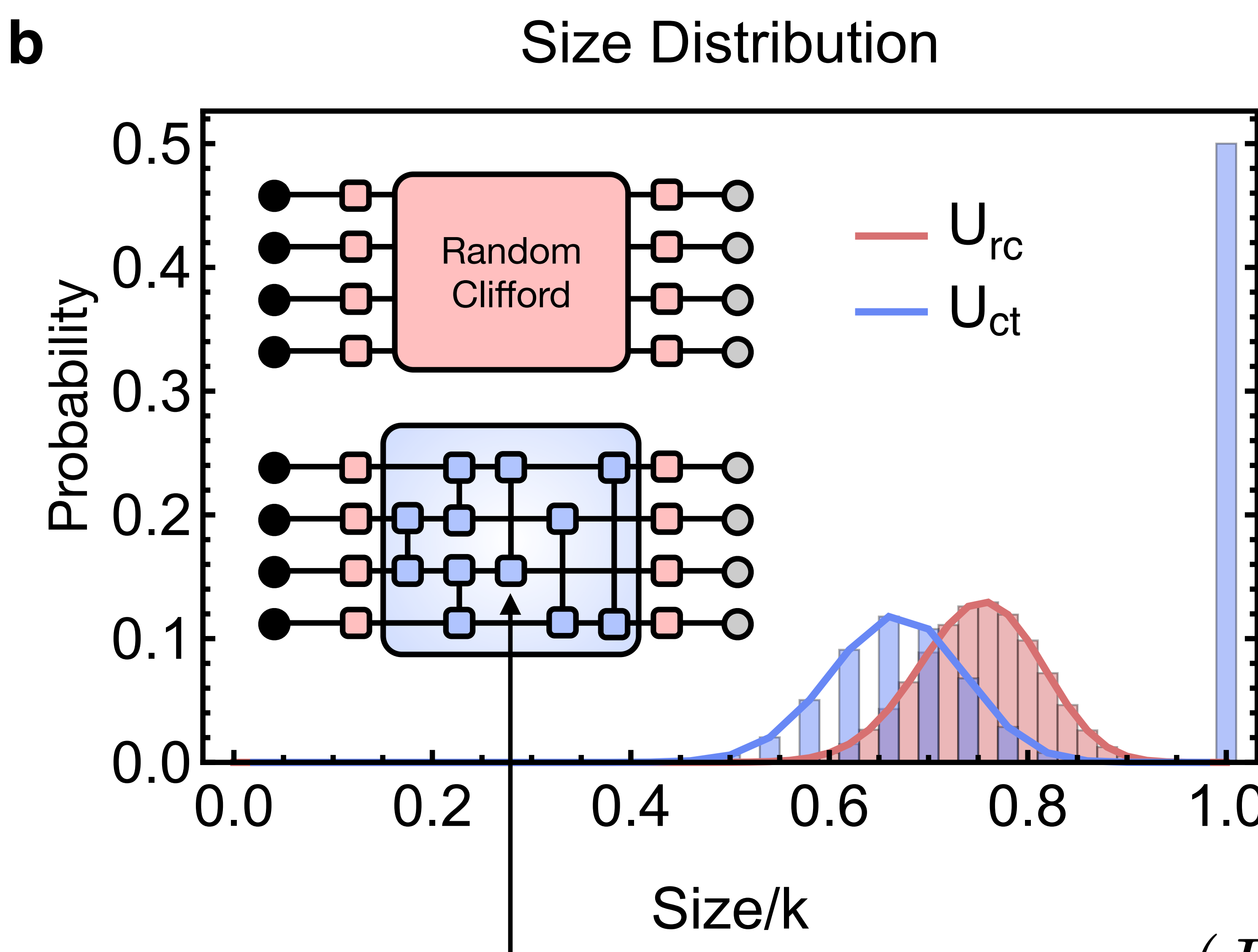
- The rule of the game is to use the unitary circuit to contract the observable size on the measurement side to reduce sample complexity



Y. Wu, C. Wang, J. Yao, H. Zhai, Y.-Z.You, P. Zhang. arXiv:2412.01850

Contractive Unitary Shadow Tomography

- Random unitary circuit are thermalizing (equilibrium), can we go out-of-equilibrium?



All-to-all $U_{ij} = \exp(\frac{i\pi}{4} Z_i Z_j)$

田忌赛马 strategy: scarify the efficiency for half of the observables to trade for efficiency of the other half.

$$w_{ct}(P) = \frac{1}{2}\frac{1}{3^k} + \frac{1}{2}\left(\frac{5}{9}\right)^k$$

Y. Wu, C. Wang, J. Yao, H. Zhai, Y.-Z.You, P. Zhang. arXiv:2412.01850

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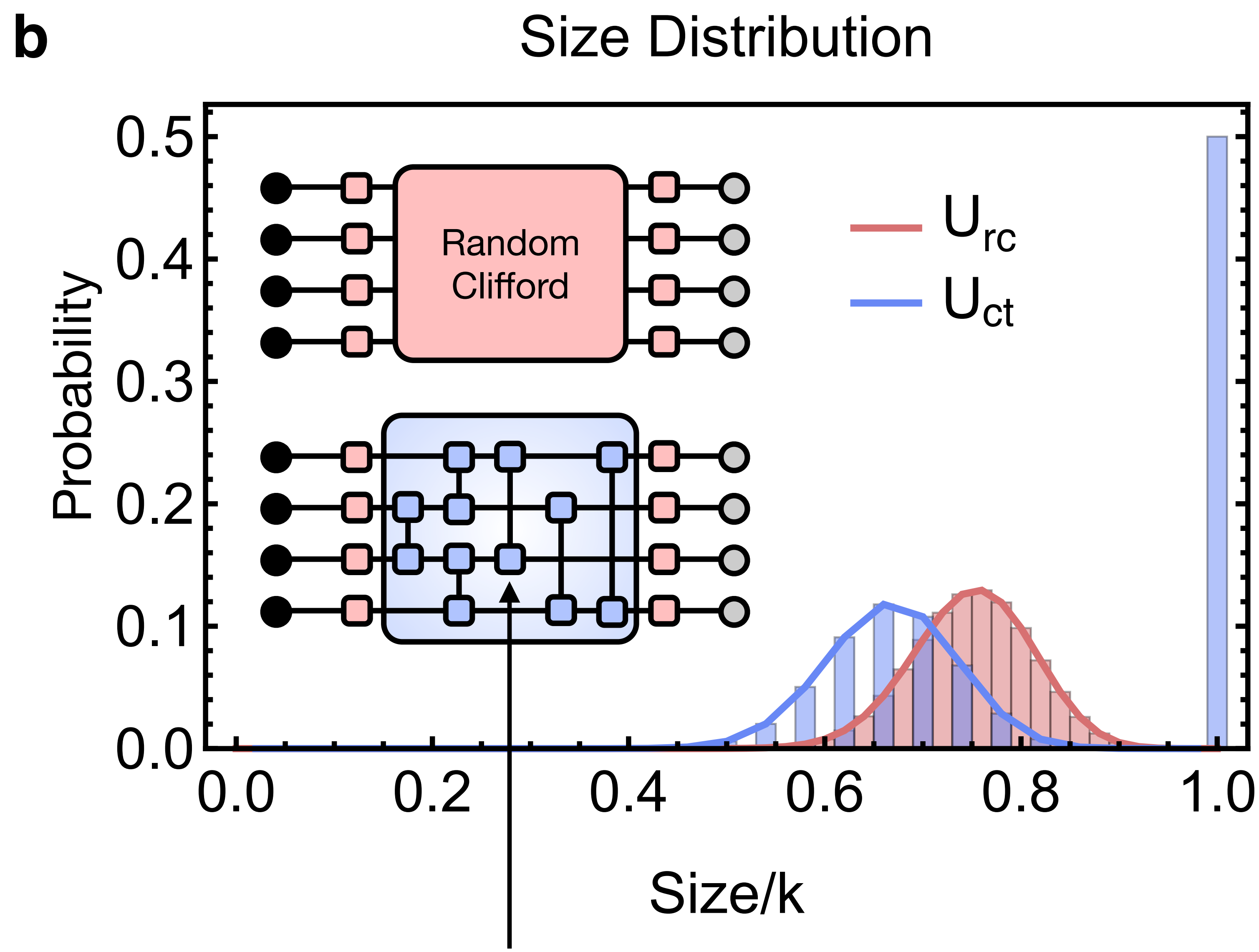

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田忌赛马 strategy: scarify the efficiency for half of the observables to trade for efficiency of the other half.

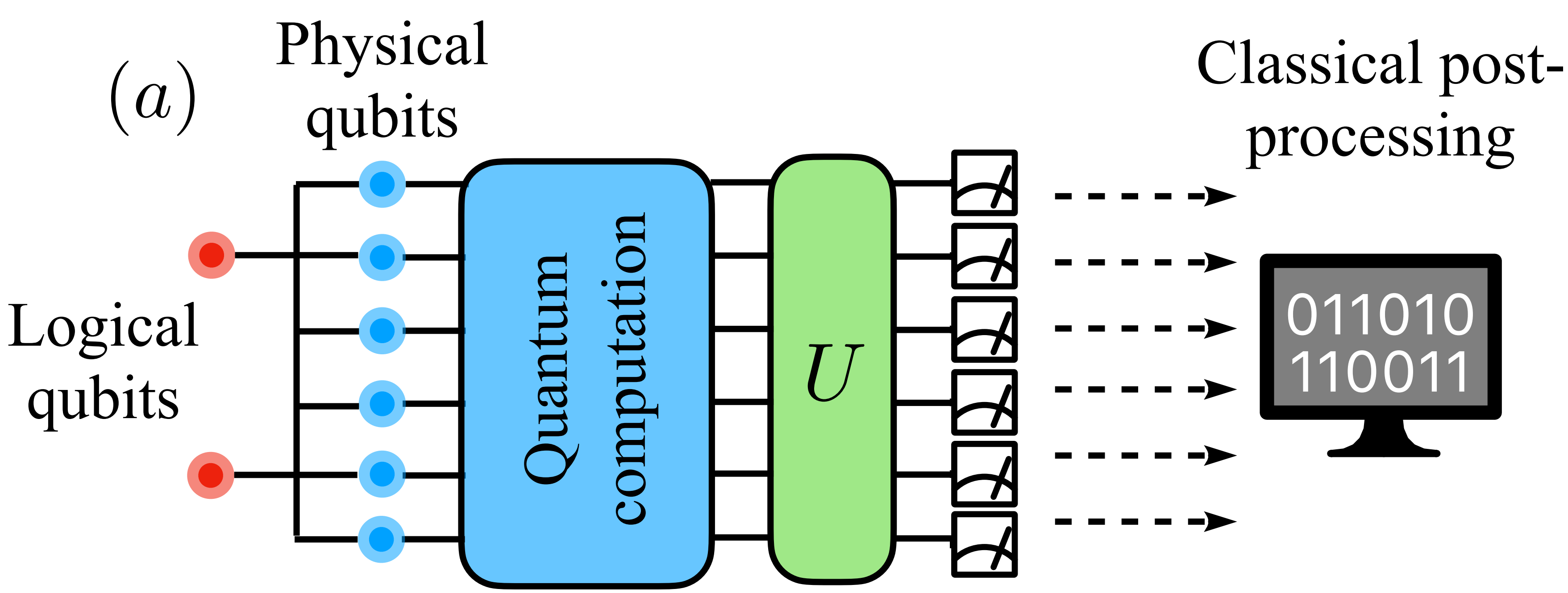
$$\boxed{\|P\|^2_{\text{ct}} \sim 1.8^k}$$

Y. Wu, C. Wang, J. Yao, H. Zhai, Y.-Z.You, P. Zhang. arXiv:2412.01850

Quantum Error Mitigation

- Classical shadow tomography has many amazing applications, one example is to implement code subspace projection in quantum error mitigation (QEM)

Hu, LaRose, You, Rieffel, Wang (2022); Seif, Cian, Zhou, Chen, Jiang (2022)



Code Projection: $\langle O \rangle_{\mathrm{QEM}} = \dfrac{\mathrm{Tr}(\Pi\rho\Pi^\dagger O)}{\mathrm{Tr}(\Pi\rho\Pi^\dagger)}$

$$\mathrm{Tr}(\Pi\rho\Pi^\dagger O) = \mathop{\mathbb{E}}_{\hat{\sigma}\in\mathcal{E}_{\sigma|\rho}} \mathrm{Tr}(\mathcal{M}^{-1}[\hat{\sigma}]\Pi^\dagger O \Pi)$$

Quantum Error Mitigation

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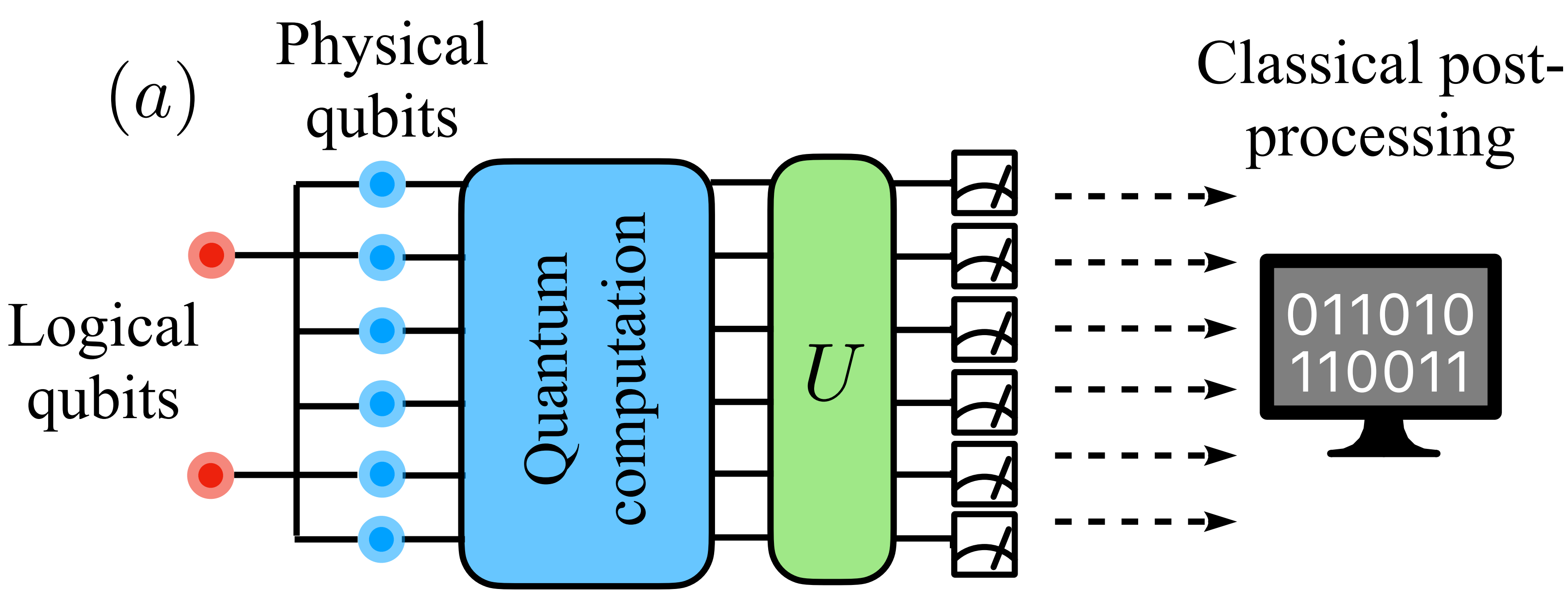
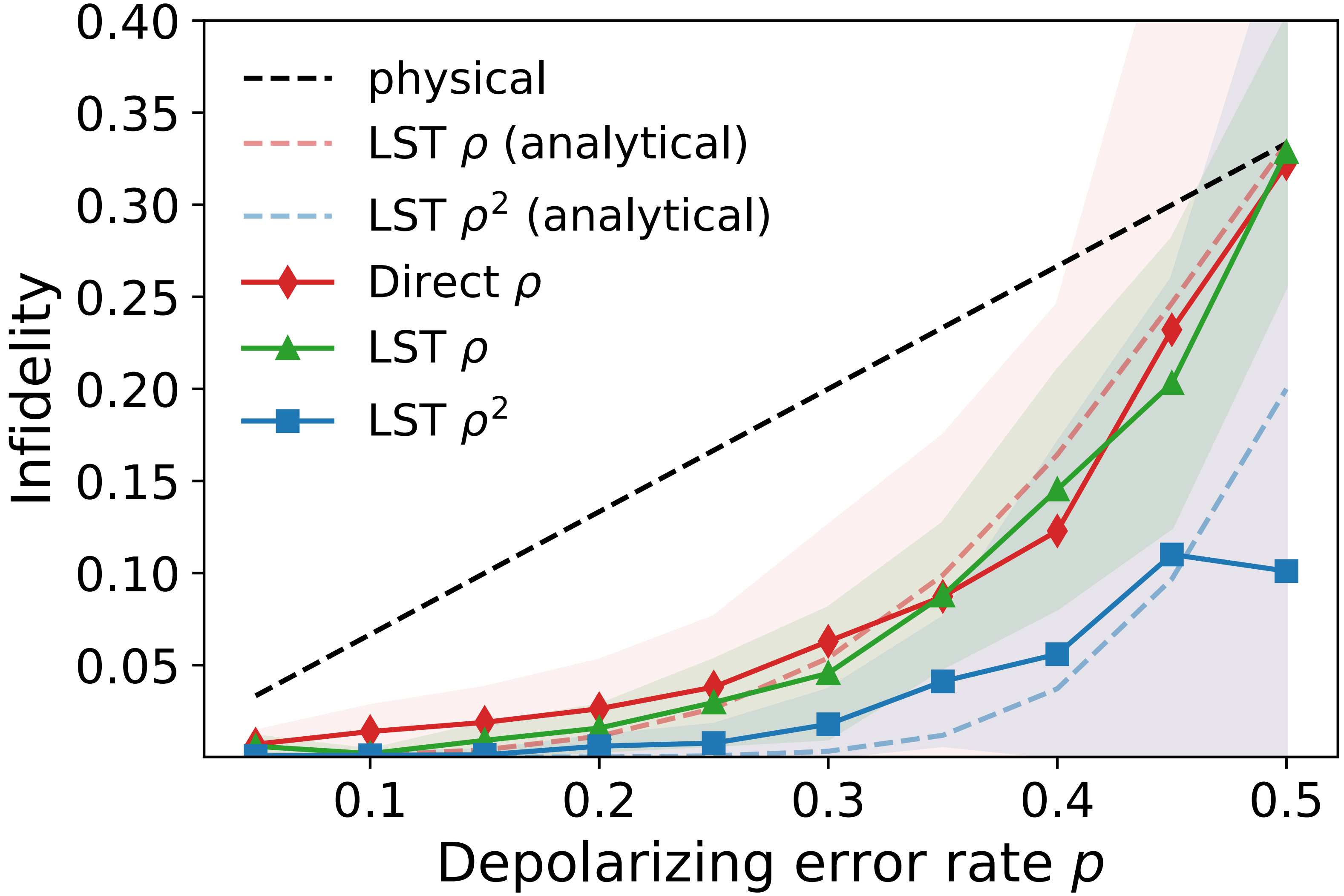


Virtual Cooling: $$\langle O \rangle_{\mathrm{QEM}} = \frac{\mathrm{Tr}(\rho^2 O)}{\mathrm{Tr}(\rho^2)}$$

$$\mathrm{Tr}(\rho^2 O) = \mathop{\mathbb{E}}_{\hat{\sigma},\hat{\sigma}' \in \mathcal{E}_{\sigma|\rho}} \mathrm{Tr}(\mathcal{M}^{-1}[\hat{\sigma}]\mathcal{M}^{-1}[\hat{\sigma}']O)$$

Quantum Error Mitigation

- Assume each physical bit is subject to depolarization error
- QEM can reduce the infidelity when the error rate is small
- **Logical shadow tomography** (LST) demonstrates superior sample efficiency (small variance)



H.-Y. Hu, R. LaRose, Y.-Z. You, E. Rieffel, Z. Wang. arXiv:2203.07263 (2022)

Experimental Realization

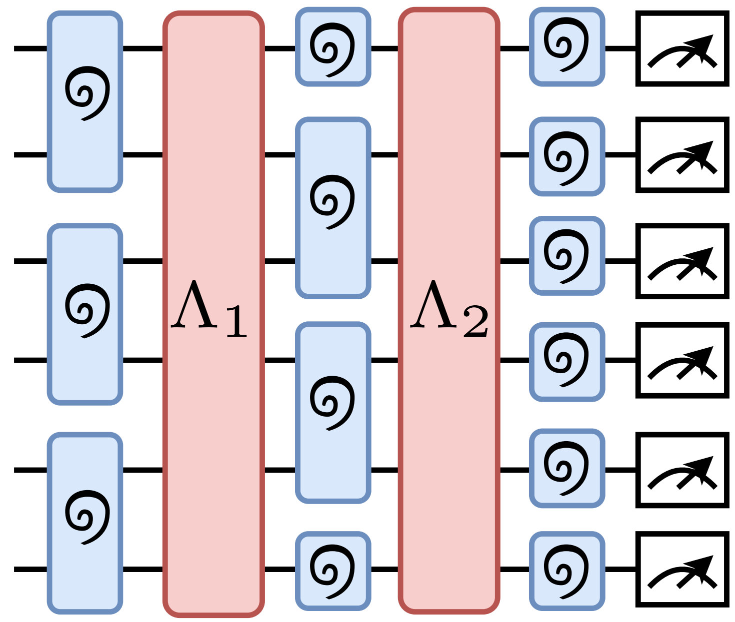






Noise-Resilient Shallow Shadow

- Challenge: real quantum devices are noisy



$$\mathscr{M}_\Lambda[\rho] = \sum_{\mathbf{b}} \mathop{\mathbb{E}}_{U\sim p(U)} \langle \mathbf{b} | \mathscr{C}_{U,\Lambda}[\rho] | \mathbf{b} \rangle (U^\dagger | \mathbf{b} \rangle \langle \mathbf{b} | U)$$

Noisy measure *Ideal prepare*

- Solution: As long as each unitary gate is locally scrambled (twirled) $\mathscr{M}_\Lambda[P] = \omega_\Lambda(P)P$ is still diagonal and $\omega_\Lambda(P)$ is still a Markov process

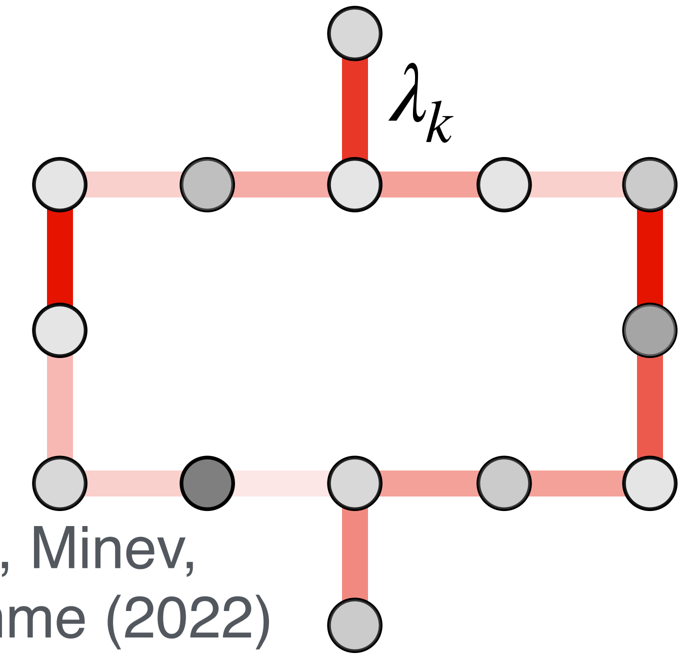
Hu, Gu, Majumder et.al. 2402.17911

Noise-Resilient Shallow Shadow

- Challenge: what is the noise model?

$$\mathscr{C}_{U,\Lambda}[\rho] = \mathrm{e}^{\mathscr{L}}[U\rho U^\dagger]$$

where $\mathscr{L}[\rho] = \sum_k \lambda_k (P_k \rho P_k - \rho)$



van den Berg, Minev, Kandala, Temme (2022)

- Solution: randomized benchmarking
  - Observable expectation values should be predicted by

$$\mathrm{Tr}(\rho P) = \frac{1}{\omega(P)?} \mathop{\mathbb{E}}_{\sigma \sim p(\sigma|\rho)} \mathrm{Tr}(\sigma P)$$

  - Suppose Pauli weight is unknown — Infer Pauli weights from a known state (e.g. $\rho = |\mathbf{0}\rangle\langle\mathbf{0}|$)

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Noise-Resilient Shallow Shadow

- Challenge: what is the noise model?

$$\mathscr{C}_{U,\Lambda}[\rho] = \mathrm{e}^{\mathscr{L}}[U\rho U^\dagger]$$

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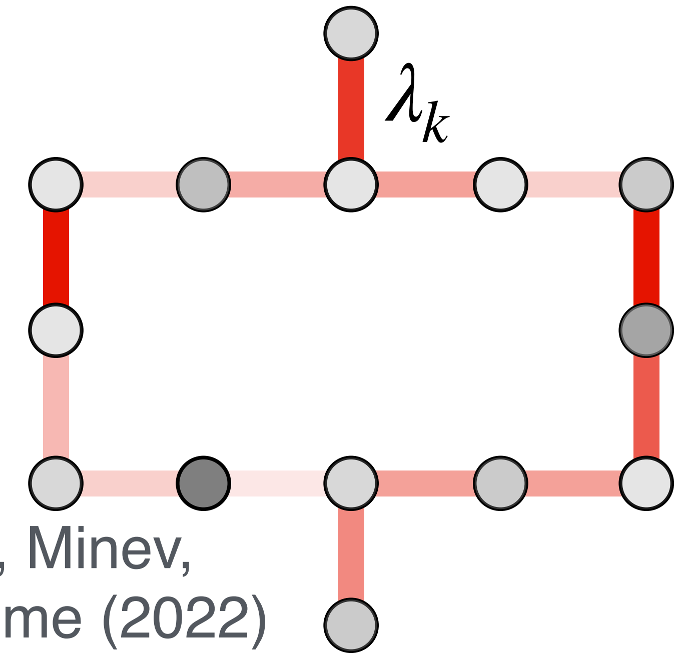

van den Berg, Minev, Kandala, Temme (2022)

- Solution: randomized benchmarking
  - Measure Pauli weights on a known state (e.g. $\rho = |\mathbf{0}\rangle\langle\mathbf{0}|$)

$$\omega_{\text{data}}(P) = \frac{1}{\mathrm{Tr}(\rho P)} \mathop{\mathbb{E}}_{\sigma \sim p(\sigma|\rho)} \mathrm{Tr}(\sigma P)$$

  - Calculate Pauli weights $\omega_\Lambda(P)$ assuming λ_k parameters
  - Determine λ_k parameters by $\min \|\omega_\Lambda - \omega_{\text{data}}\|^2$

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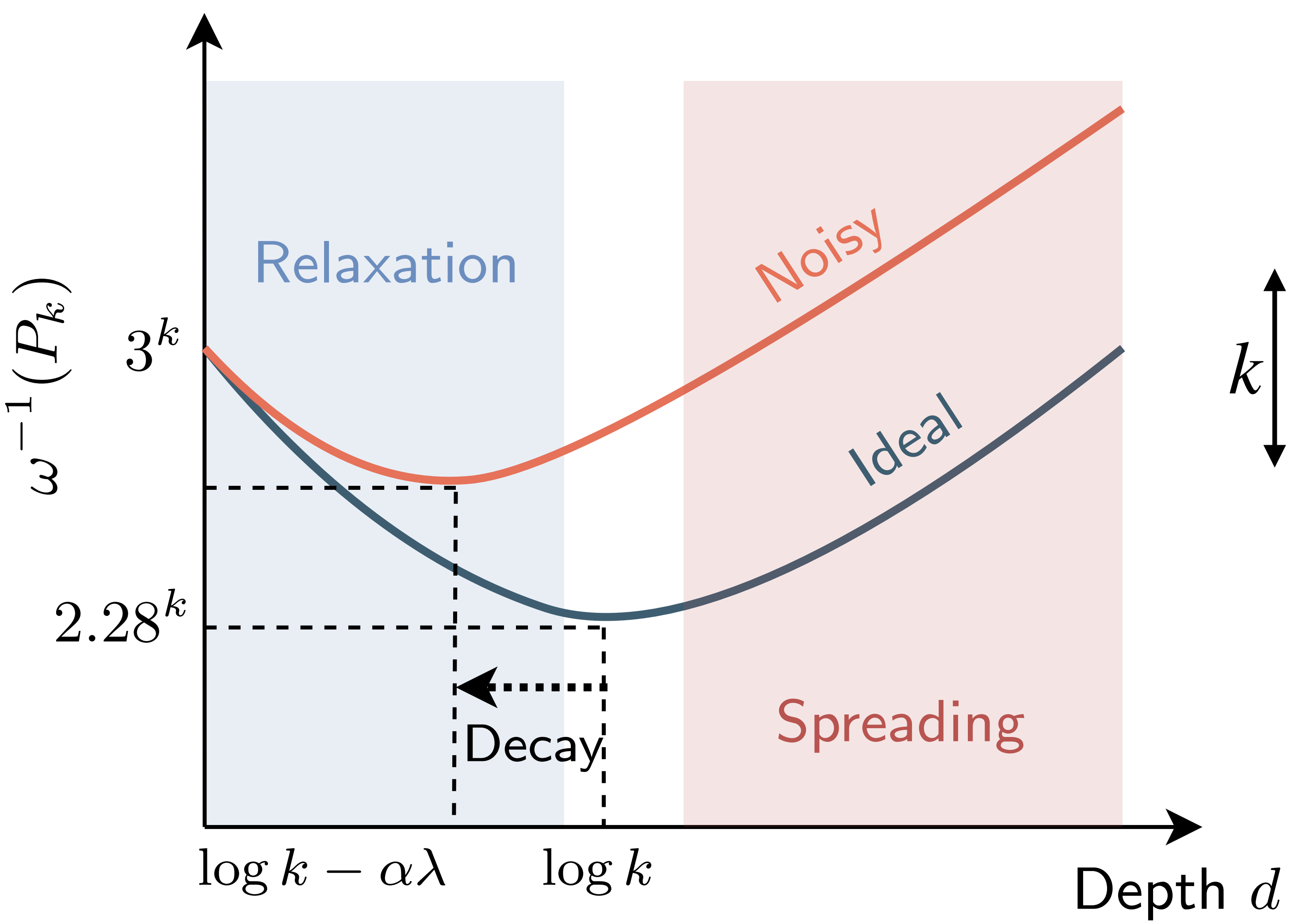
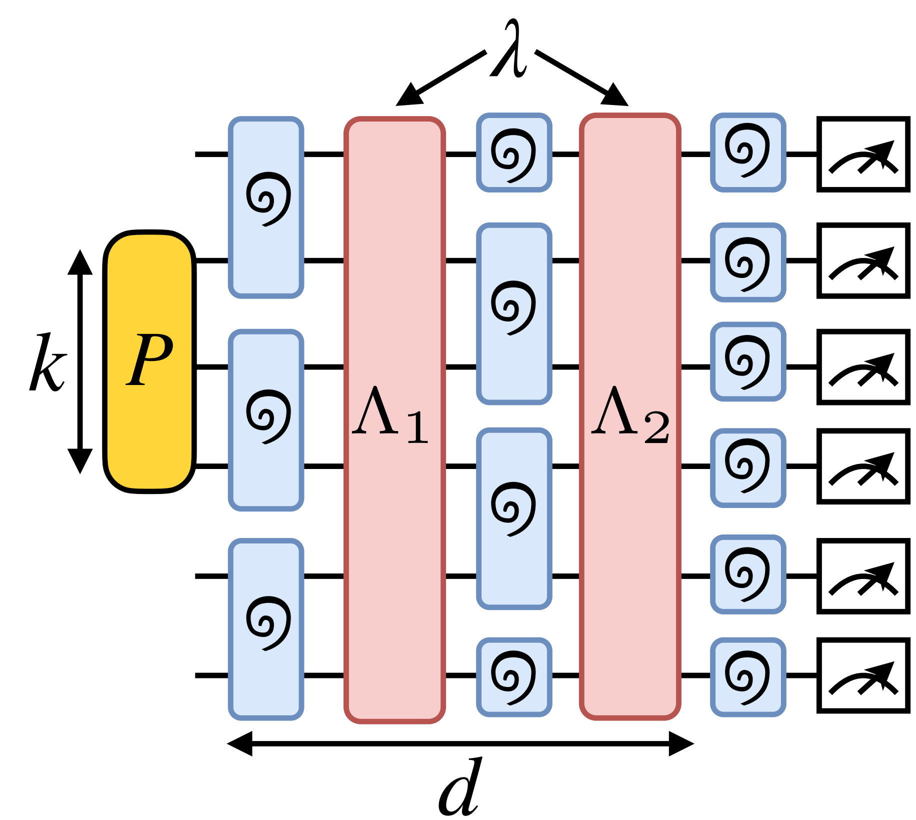
Variance-Bias Tradeoff

- Assuming single-qubit depolarization noise of strength λ,

$$\log_3 \|P\|_{\mathsf{sh}}^2 \leq (k+d)\left(\frac{3}{4} + \frac{(4/5)^{2d}}{d^{3/2}} + \frac{d\lambda}{\log 3}\right)$$

Sample Complexity Upper Bound





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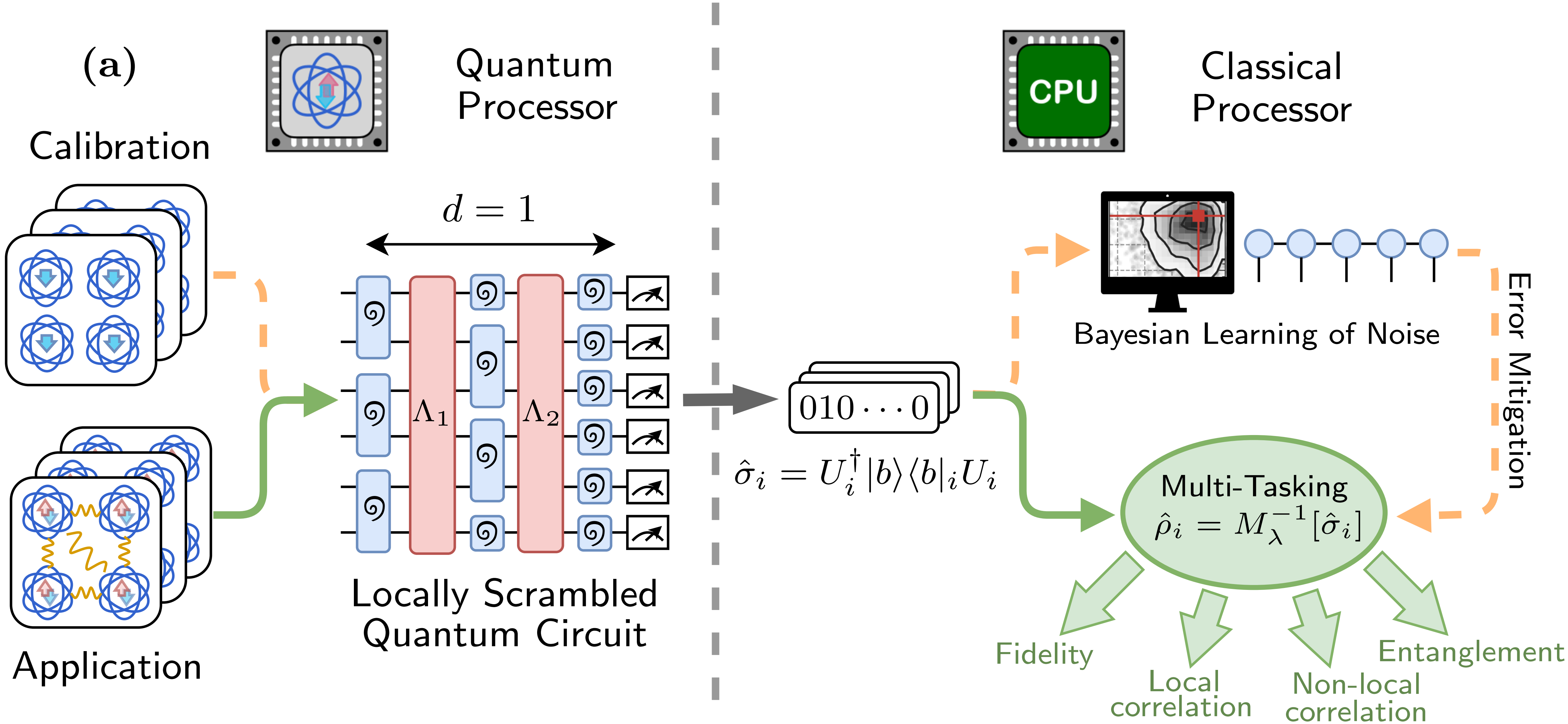
Noise-Resilient Shallow Shadow

- Overview of robust shallow shadow protocol



- Experiment: 18 qubits on a 127-qubit superconducting quantum computer (ibm_kyiv), 10k random circuits in 6 min

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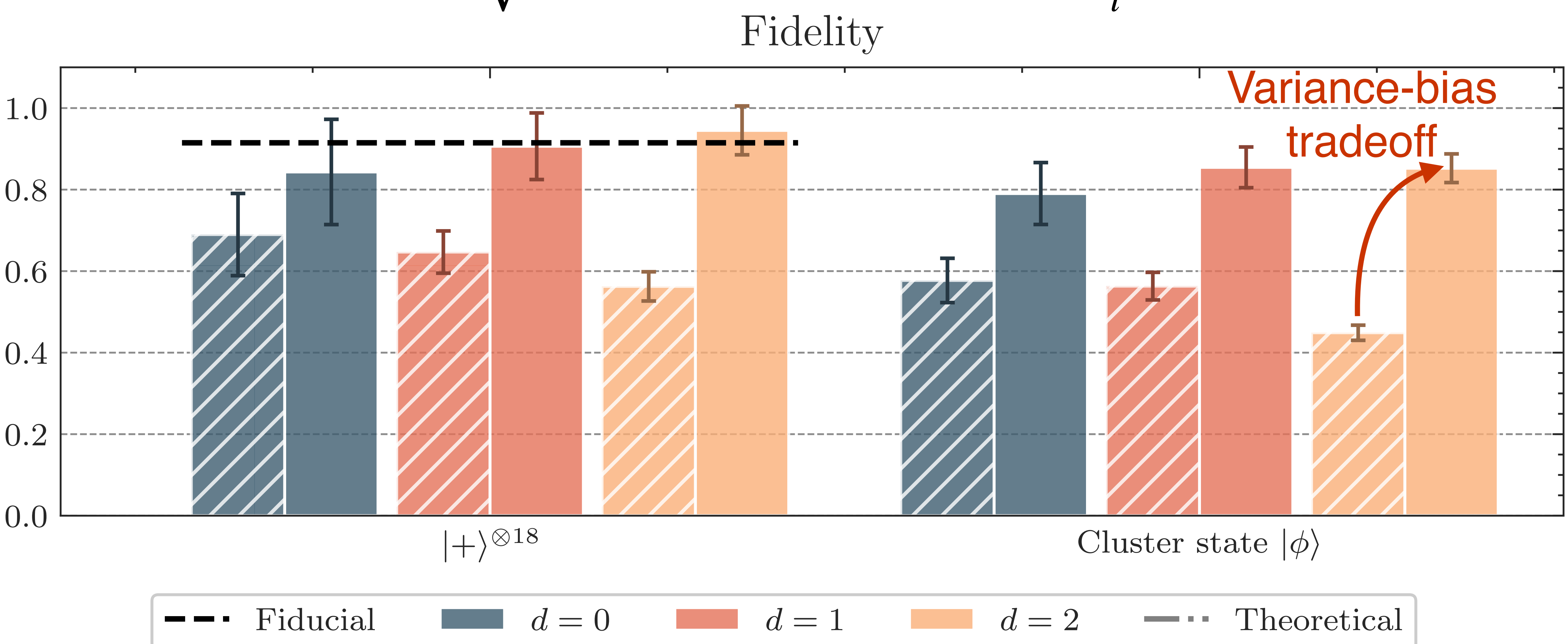
Fidelity Estimation

- Test states: plus state & cluster state

$$|+\rangle^{\otimes N} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)^{\otimes N} \qquad |\phi\rangle = \prod_{i}^{N-1} \mathrm{CZ}_{i,i+1} |+\rangle^{\otimes N}$$



- Use calibration: yes - solid, no - shaded; d - circuit depth

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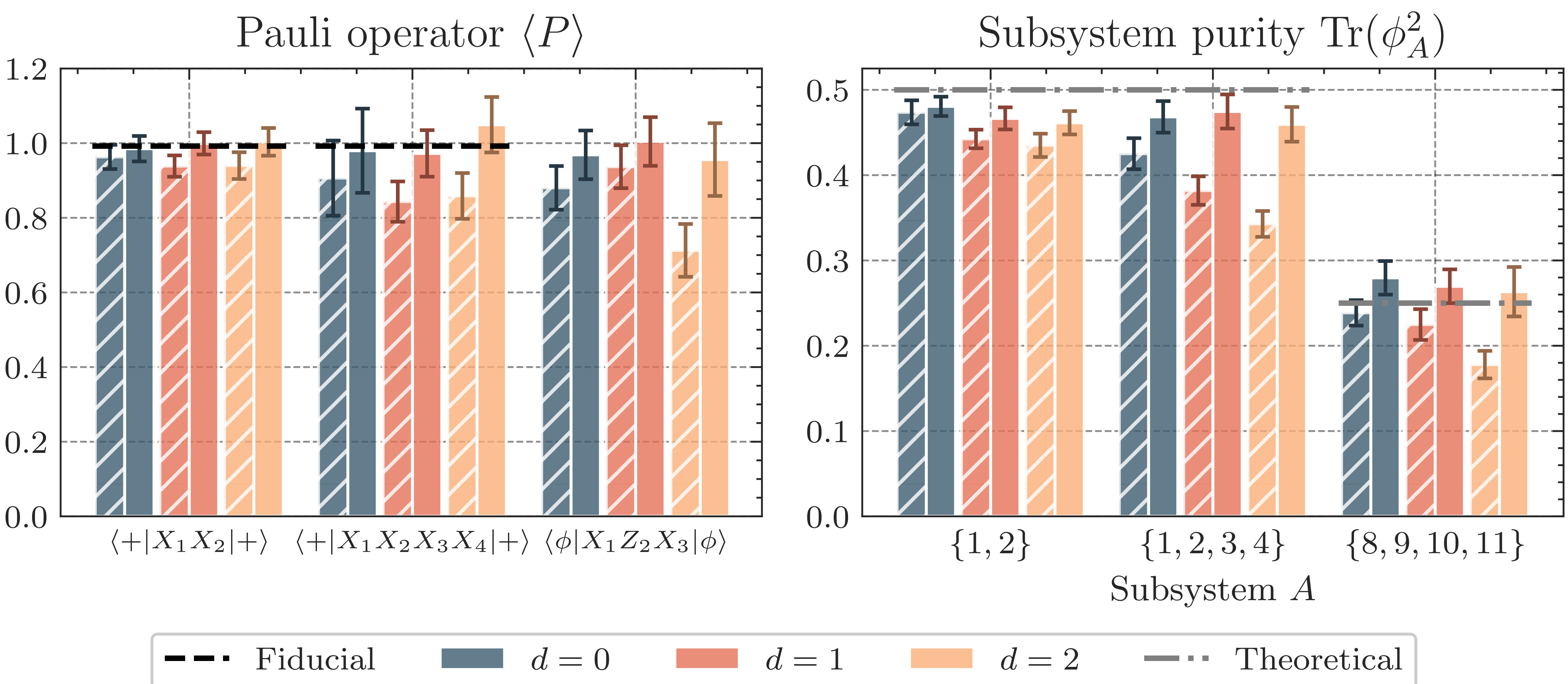
Property Predictions

- Linear and non-linear properties



- Use calibration: yes - solid, no - shaded; d - circuit depth

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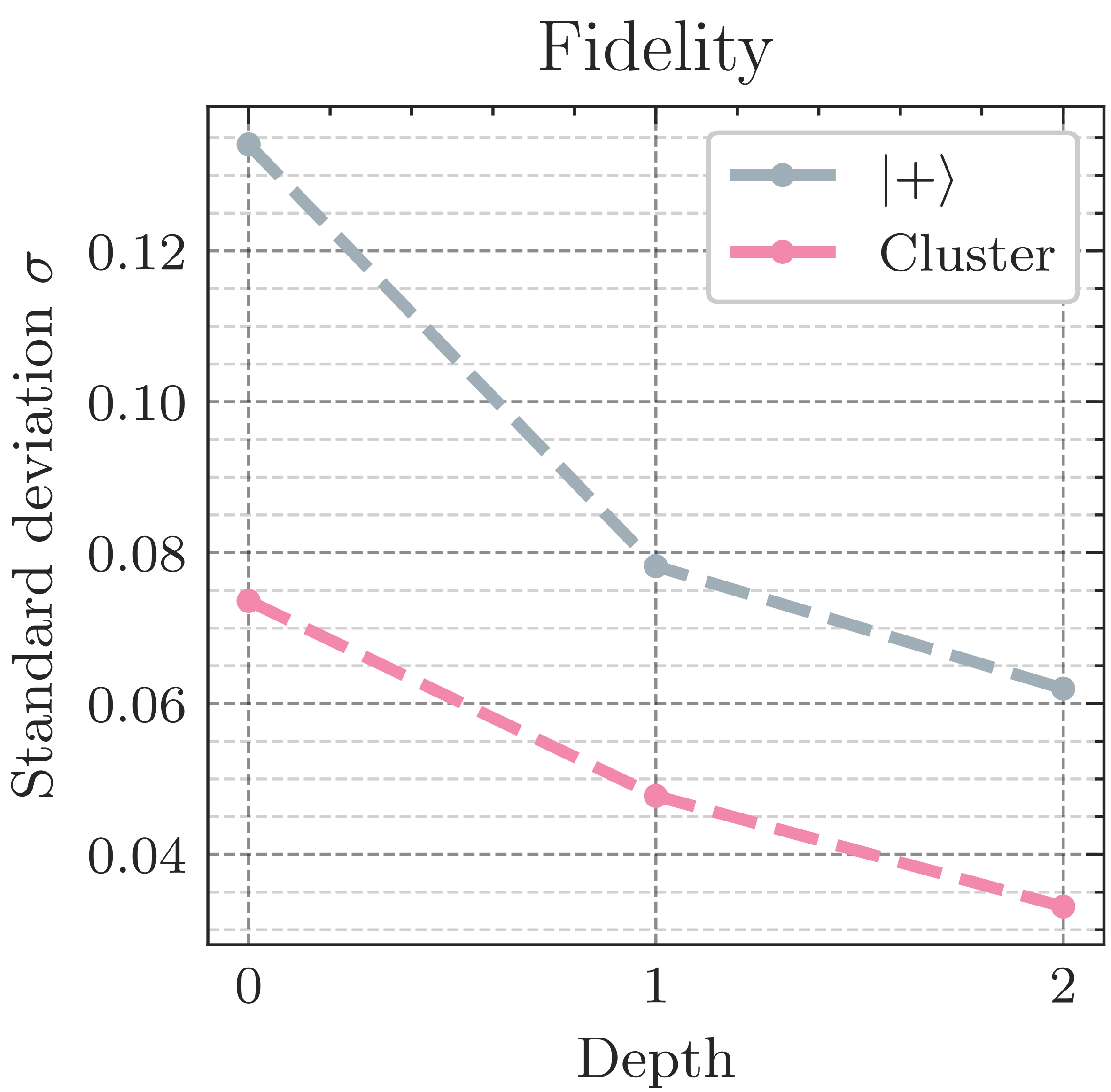
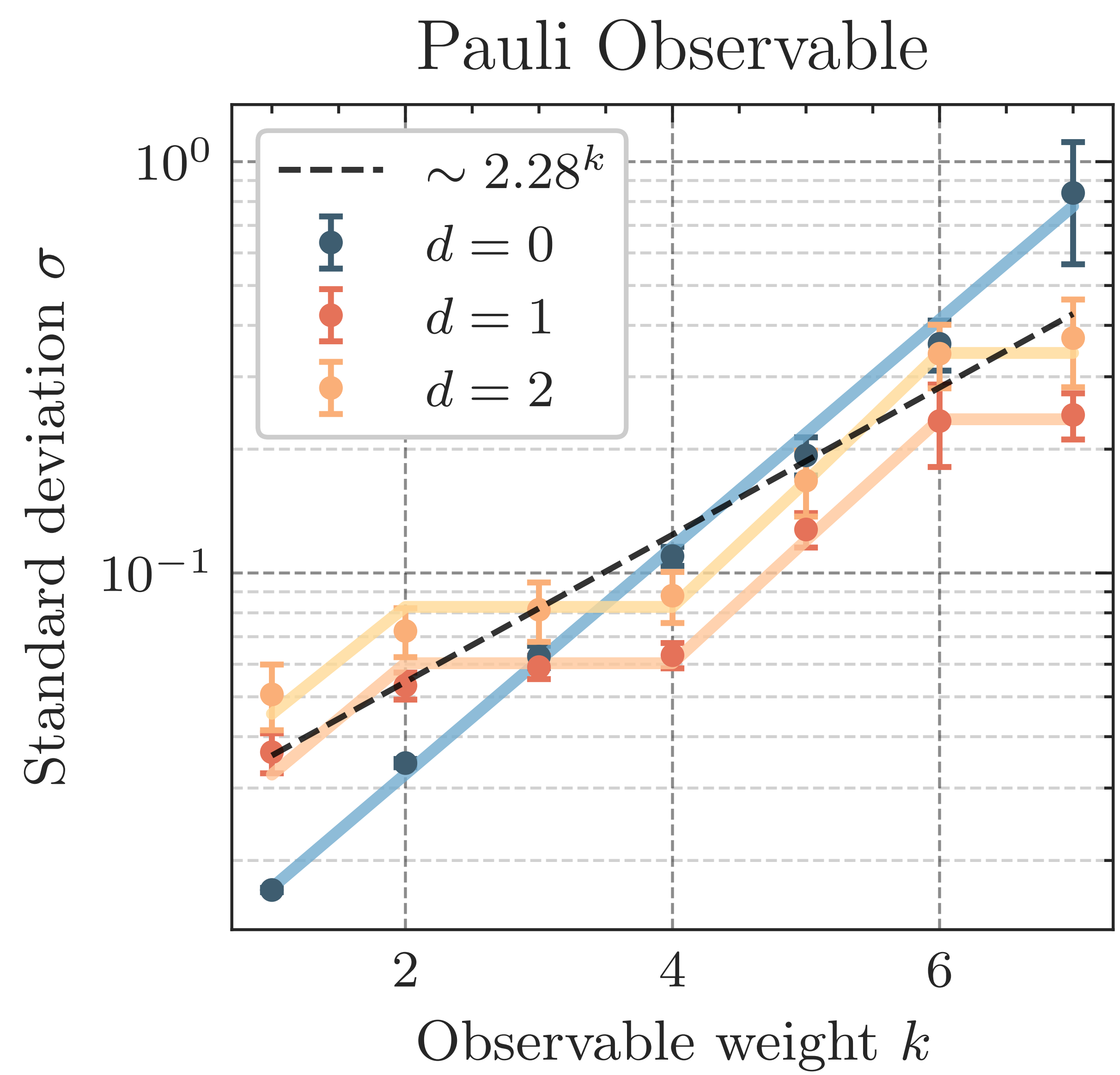
Standard Deviation Scaling

- Advantage of shallow shadow: reduced/optimized standard deviation with a few layers of twirled CNOT gates





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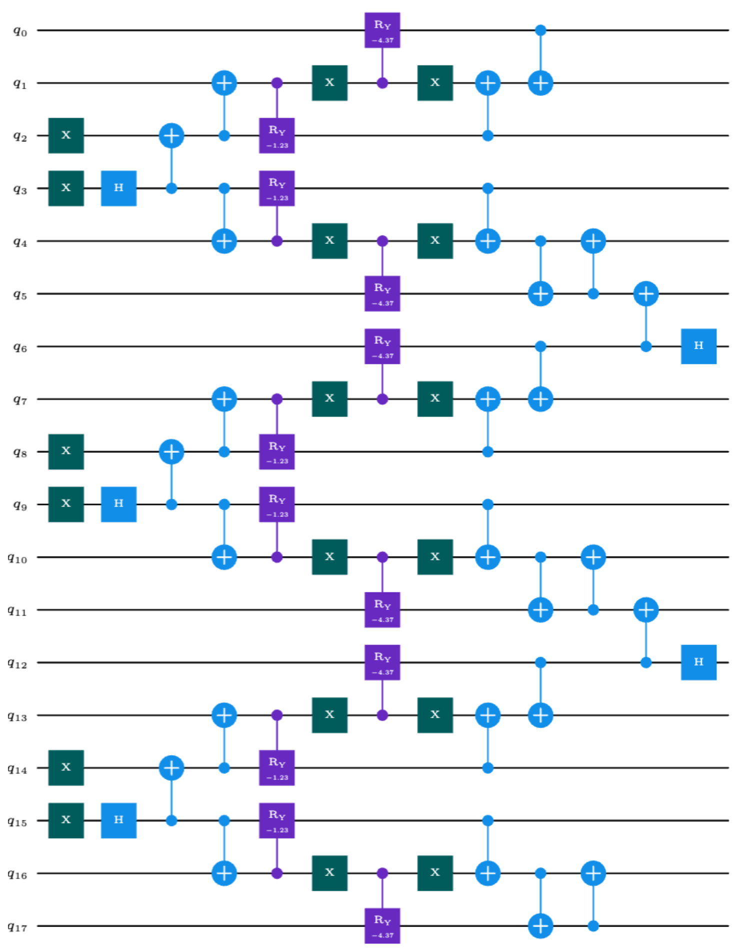
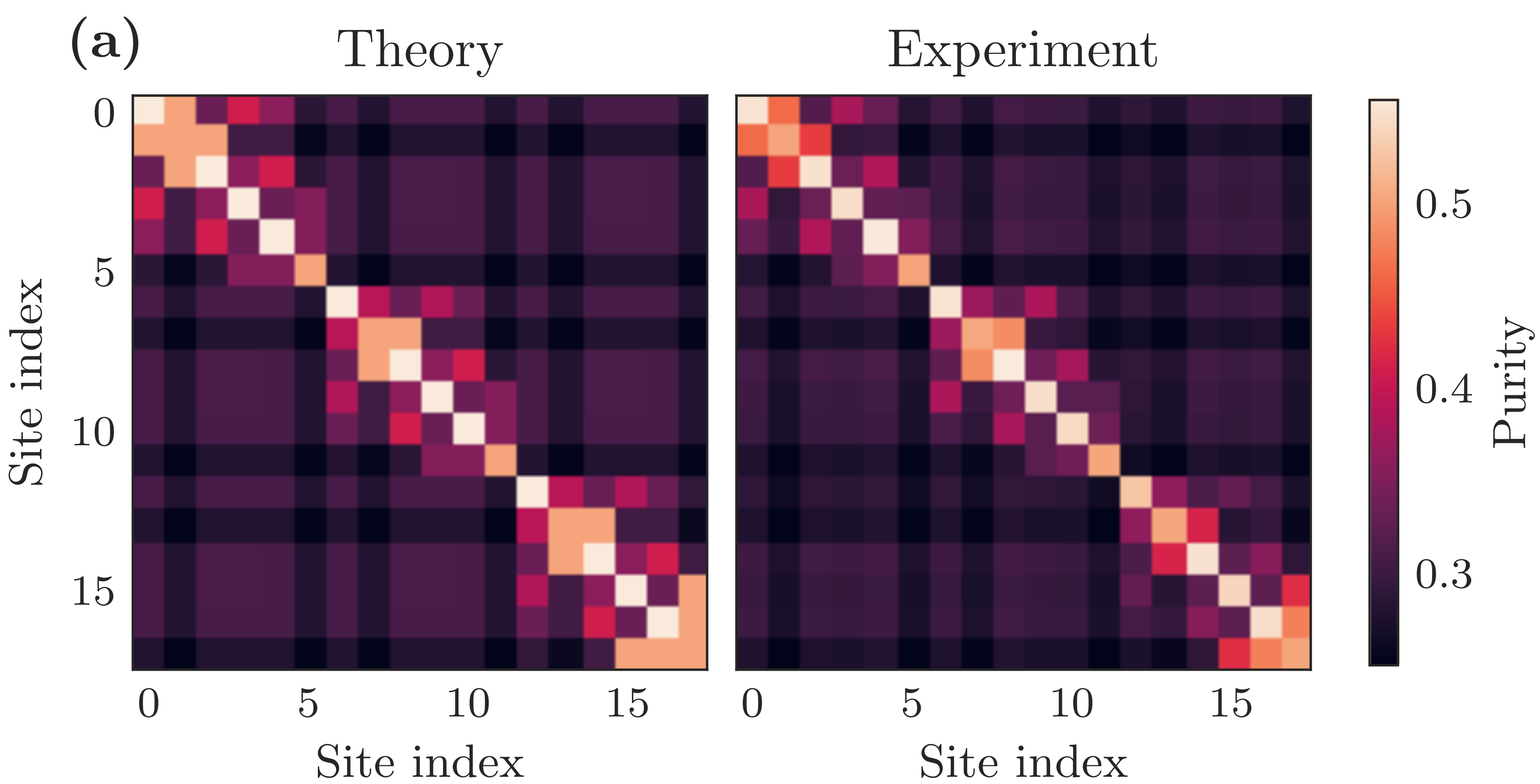
Beyond Stabilizer State

- Measure AKLT resource state, predict two-site purity Tr ρ_{ij}^2



Smith, Crane, Wiebe, Girvin (2022)



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Summary

- With locally-scrambled quantum dynamics, we extend the classical shadow tomography to a large class of quantum circuits, which is
  - Scalable (efficient classical post-processing)
  - Flexible (arbitrary circuit structure / quantum dynamics)
  - NISQ friendly (shallow circuits, simple gates, available devices)
- We expect our approach to have broad applications in many quantum information processing tasks (e.g. quantum error mitigation)

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Thanks for your attention!