Symmetric Mass Generation

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[1] Juven Wang, Yi-Zhuang You, Symmetric Mass Generation. *Symmetry* **2022**, *14*(7), 1475 (arXiv: 2204.14271)

Paths to Quantum Field Theory 2022
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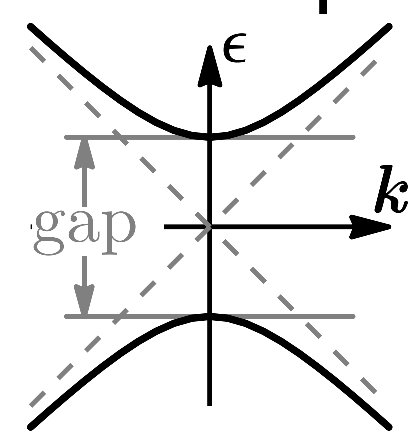
Fermion Bilinear Mass

- Relativistic fermion in d-dimensional spacetime

$$S[\psi] = \int \mathrm{d}^d x\; \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi$$

  $\bar{\psi} = \psi^\dagger \gamma^0$ for Dirac fermions (or $\psi^\intercal \gamma^0$ for Majorana fermions in real representation)

- Fermion **bilinear mass** term $m\bar{\psi}\psi$
  - It creates an **energy gap** in the fermion excitation spectrum (as seen from the equation of motion)



$$(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\Rightarrow k_\mu k^\mu - m^2 = 0 \Leftrightarrow \epsilon = \pm\sqrt{\boldsymbol{k}^2 + m^2}$$

  - Fermion correlation: **short-ranged** (exponential decay)

$$\langle \bar{\psi}(0)\psi(x)\rangle \sim \mathrm{e}^{-|x|/\xi} \qquad (\xi \sim m^{-1})$$

Finite correlation length

Definition of Fermion Mass

- How to define fermion mass beyond the free-fermion limit?
  - Can not easily solve the equation of motion in the presence of fermion interaction …
  - The notion of quasi-particle may not even be well-defined under interaction.
- However, the fermion (two-point) correlation function is still well-defined by the path integral

$$\langle\bar{\psi}(0)\psi(x)\rangle = \frac{1}{Z}\int \mathcal{D}[\psi]\bar{\psi}(0)\psi(x)\mathrm{e}^{\mathrm{i}S[\psi]}$$

  - Fermion mass = inverse correlation length (~ the fermionic excitation gap in the many-body spectrum)

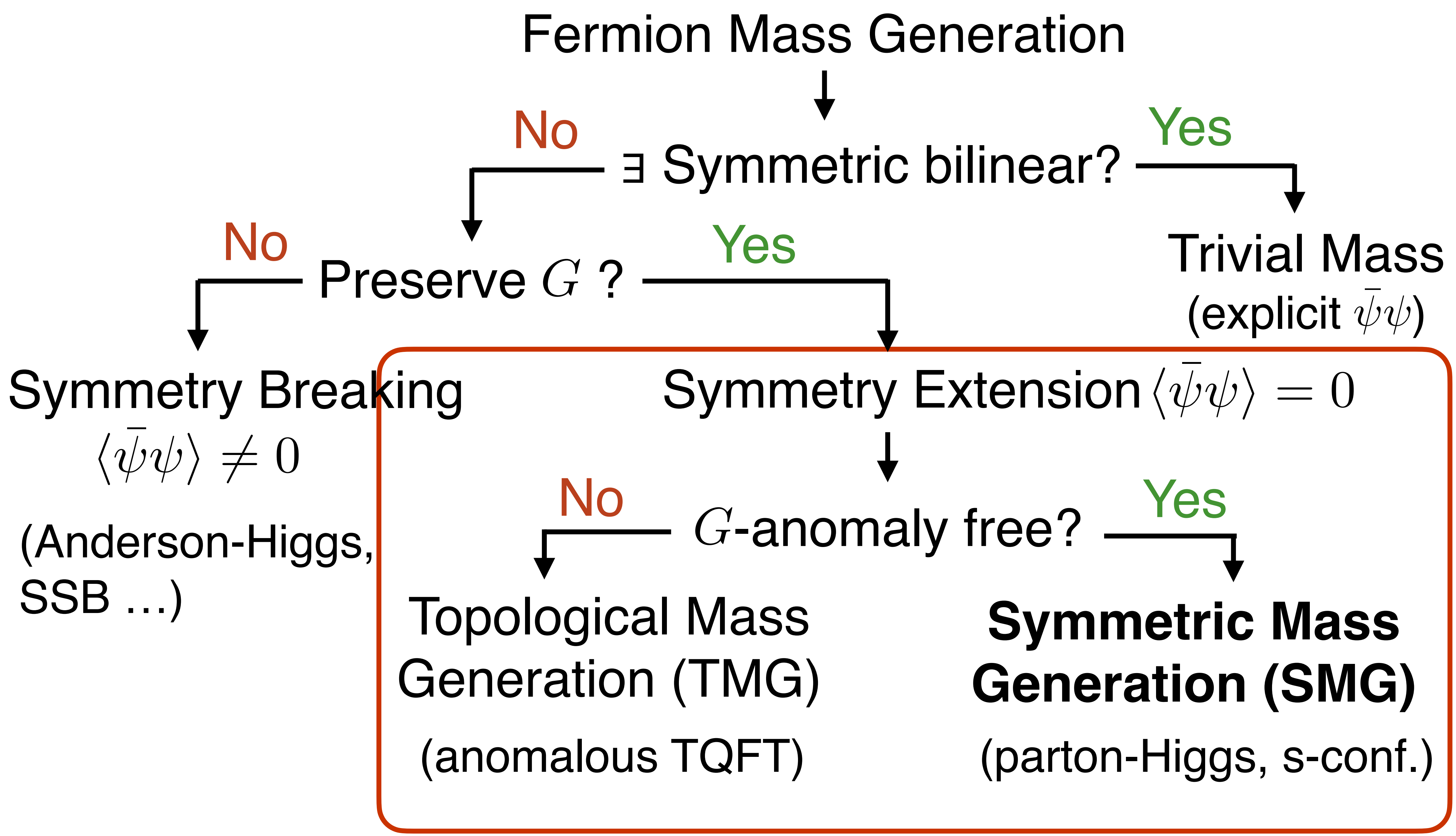
$$\langle\bar{\psi}(0)\psi(x)\rangle \sim \begin{cases} |x|^{-2\Delta_\psi} & \Rightarrow \quad m = 0 \\ \mathrm{e}^{-|x|/\xi} & \Rightarrow \quad m = 1/\xi \end{cases}$$

Fermion Mass Generation

- Fermion mass generation: How to create an excitation gap for gapless fermions?
- Higgs mechanism: condense a fermion bilinear mass
  - Involves spontaneous symmetry breaking or gauge Higgsing
  - Examples:
    - BCS superconductor $\mathrm{U}(1) \to \mathbb{Z}_2$
    - Electroweak Higgs $\mathrm{SU}(2) \times \mathrm{U}(1)_{\mathrm{Y}} \to \mathrm{U}(1)_{\mathrm{EM}}$
- This may not be the full story of fermion mass generation.
- Can fermions acquire a mass/gap without SSB?
  - Mott insulator: gap opening by charge repulsion
  - Kondo insulator: gap opening by Kondo (spin) interaction

Fermion Mass Generation

- Symmetry group G and the representation $\mathbf{r}_\psi^G$ of fermion field



Fidkowski-Kitaev Model: SMG in (0+1)D

- The first and simplest example of SMG was discovered by Fidkowski and Kitaev in 2009 — Fidkowski, Kitaev 0904.2197, 1008.4138
- A collection of Majorana fermions in (0+1)D spacetime
  - Majorana fermion operator χ_a, satisfying Clifford algebra

$$\{\chi_a, \chi_b\} = 2\delta_{ab} \quad (a, b = 1, 2, \cdots)$$

- They are Hermitian operators $\chi_a^\dagger = \chi_a$
- They can be viewed as "real/imaginary parts" of fermion creation/annihilation operators, e.g.

$$c_1 = \tfrac{1}{2}(\chi_1 + \mathrm{i}\chi_2) \quad c_1^\dagger = \tfrac{1}{2}(\chi_1 - \mathrm{i}\chi_2)$$

- Fermion number operator

$$n_1 = c_1^\dagger c_1 = \tfrac{1}{2}(1 + \mathrm{i}\chi_1\chi_2)$$

Fidkowski-Kitaev Model: SMG in (0+1)D

- A collection of Majorana fermions in (0+1)D spacetime
  - Majorana fermion operator χ_a
  - Subject to a time-reversal symmetry (anti-unitary)

$$\mathbb{Z}_2^T : \chi_a \to \chi_a, \mathrm{i} \to -\mathrm{i}$$

  and the fermion parity symmetry (unitary)

$$\mathbb{Z}_2^F : \chi_a \to -\chi_a$$

  - What could be the Hamiltonian operator H for this quantum system, preserving the $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry?
    - $\mathbb{Z}_2^F$: terms in H must only contain an even number of Majorana fermion operators, like

$$H = \mathrm{i}u_{ab}\chi_a\chi_b + u_{abcd}\chi_a\chi_b\chi_c\chi_d + \cdots \quad (u_{\dots} \in \mathbb{R})$$

Fidkowski-Kitaev Model: SMG in (0+1)D

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and the fermion parity (unitary) symmetry

$$\mathbb{Z}_2^F : \chi_a \to -\chi_a$$

  - What could be the Hamiltonian operator H for this quantum system, preserving the $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry?
    - $\mathbb{Z}_2^T$: any terms with imaginary coefficients are forbidden

$$H = \cancel{\mathrm{i}u_{ab}\chi_a\chi_b} + u_{abcd}\chi_a\chi_b\chi_c\chi_d + \cdots \quad (u_{\dots} \in \mathbb{R})$$

    - Without fermion interaction (at bilinear level): $H = 0$.

Fidkowski-Kitaev Model: SMG in (0+1)D

- Without fermion interaction (at bilinear level): $H = 0$.
  - All states in the many-body Hilbert space are degenerate → fermion “excitations” are gapless (there is no energy separation between odd and even fermion parity states)
  - How many states are there?
    - Every pair of Majorana modes = a complex fermion mode → 2-fold degeneracy

$$
\begin{aligned}
n_1 &= c_1^\dagger c_1 = \tfrac{1}{2}(1 + \mathrm{i}\chi_1\chi_2) \in \{0,1\} \\
n_2 &= c_2^\dagger c_2 = \tfrac{1}{2}(1 + \mathrm{i}\chi_3\chi_4) \in \{0,1\} \\
&\dots
\end{aligned}
$$

    - $2n$ Majorana modes → 2^n-dim Hilbert space
- Without interaction, this degeneracy can not be lifted → looks like a $\mathbb{Z}$-classified anomaly (but actually not)

Fidkowski-Kitaev Model: SMG in (0+1)D

- What can we do with fermion interaction?
  - Consider eight Majorana modes, grouped into four complex fermion modes

$$c_a = \frac{1}{2}(\chi_{2a-1} + \mathrm{i}\chi_{2a}) \quad (a = 1, 2, 3, 4)$$

  - Then it is possible to turn on an interaction

$$H = c_1 c_2 c_3 c_4 + c_4^\dagger c_3^\dagger c_2^\dagger c_1^\dagger$$

  that hybridizes $|0000\rangle$ and $|1111\rangle$ states (in the Fock state basis $|n_1 n_2 n_3 n_4\rangle$ labeled by fermion occupation numbers)

  - The ground state is

$$\frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle)$$

  with energy -1 (the other states have energy +1 or 0).

Fidkowski-Kitaev Model: SMG in (0+1)D

- What can we do with fermion interaction?
  - The ground state is

$$\frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle)$$

    - Unique (non-degenerated)
    - Gapped (order-one energy gap from all the remaining 15 states in the Hilbert space)
    - No fermion bilinear expectation value

$$\forall a, b : \langle \mathrm{i}\chi_a \chi_b \rangle = 0$$

    - Symmetric: preserving the $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry
- This example shows that it is possible to open an energy gap in fermion systems without any bilinear condensation → Symmetric mass generation (SMG)

Fidkowski-Kitaev Model: SMG in (0+1)D

- What is special about the number eight?
  - This is required by the anomaly cancellation.
  - (0+1)D fermions with $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry (or the Pin^- spacetime-internal symmetry, or the BDI symmetry class), has a non-perturbative global anomaly

$$\nu \in \mathrm{TP}_2(\mathrm{Pin}^-) = \mathbb{Z}_8$$

  - The anomaly index ν = the number of Majorana modes.
- With eight Majorana modes, anomaly vanishes → the system can be trivially gapped without breaking symmetry.
- However, the $\mathbb{Z}_2^T$ symmetry is still restrictive enough to forbid any bilinear masses → interaction is the only solution, and we have already seen one example of such interaction

Fidkowski-Kitaev Model: SMG in (0+1)D

- Conclusion: SMG can happen in (0+1)D
  - when there are eight Majorana fermion modes,
  - when appropriate interaction is applied.
- The SMG interaction is not unique, but also not arbitrary
  - Charge-4e interaction ($\mathrm{SU}(4)$ symmetric)

$$H = c_1 c_2 c_3 c_4 + \text{h.c.} \to \frac{1}{\sqrt{2}}(|0000\rangle - |1111\rangle)$$

  - Heisenberg interaction ($\mathrm{Spin}(4) \times_{\mathbb{Z}_2} \mathrm{SU}(2)$ symmetric)

$$H = \boldsymbol{S}_{\mathrm{I}} \cdot \boldsymbol{S}_{\mathrm{II}} \to \frac{1}{\sqrt{2}}(|1001\rangle - |0110\rangle)$$

  - They all stabilize a unique ground state with a gap to all excitations, without breaking the $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry, without any fermion bilinear expectation i.e. $\langle \mathrm{i}\chi_a \chi_b \rangle = 0$

Fidkowski-Kitaev Model: SMG in (0+1)D

- Conclusion: SMG can happen in (0+1)D
  - when there are eight Majorana fermion modes,
  - when appropriate interaction is applied.
- The SMG interaction is not unique, but also not arbitrary
  - However, the following interaction will not work

$$H = \chi_1\chi_2\chi_3\chi_4 + \chi_5\chi_6\chi_7\chi_8 \rightarrow |0000\rangle$$

$$|1100\rangle$$

$$|0011\rangle$$

$$|1111\rangle$$

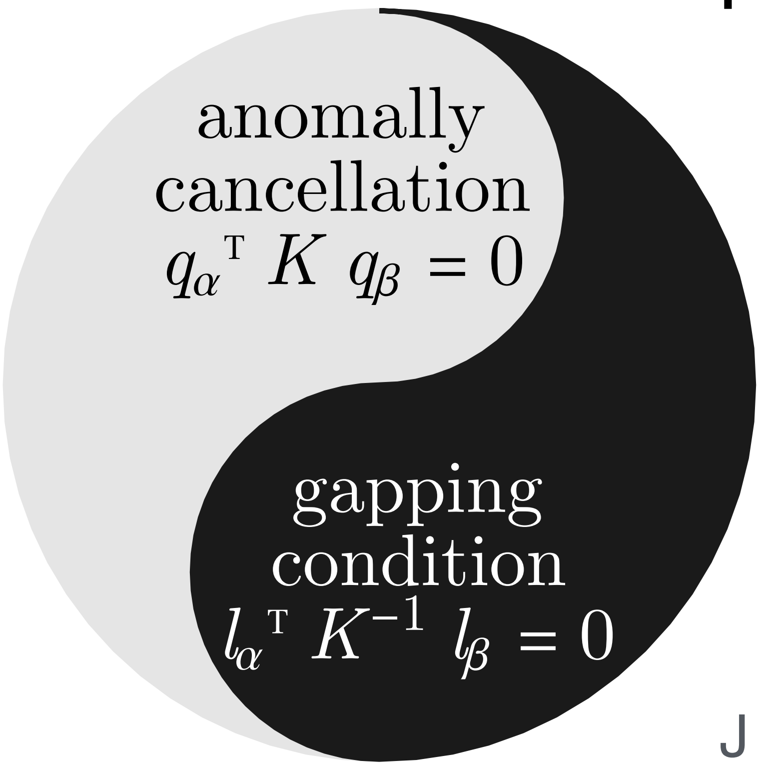
Preserves all required symmetries

But ground states are still degenerated

- How do we know if an interaction works or not? What is the gapping criterion for a proposed interaction?

3-4-5-0 Model: SMG in (1+1)D

- What is the designing principle of the SMG interaction?
  - Kinematics - anomaly cancellation:
    the fermion system must be free from *any* anomaly
  - Dynamics - gapping condition:
    there *exist* interactions to drive the system to a new RG fixed point with all low-energy freedoms trivially gapped
- In (1+1)D, these two conditions are equivalent to each other



J Wang, 2207.14813

3-4-5-0 Model: SMG in (1+1)D

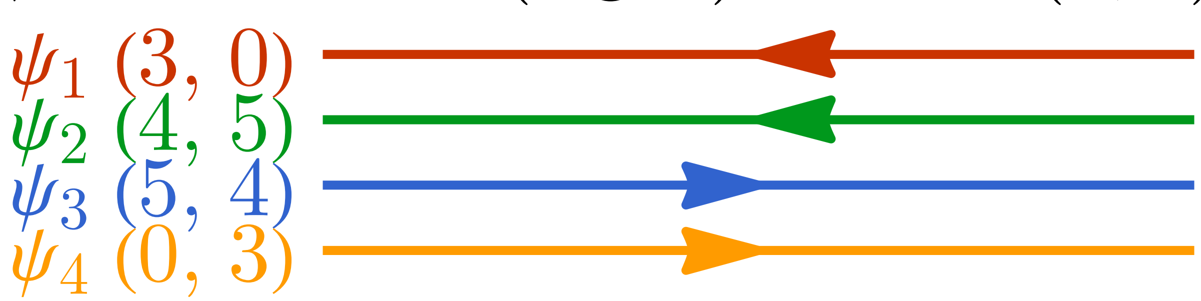
- 3-4-5-0 chiral fermions in (1+1)D

$$\mathcal{L} = \sum_{a=1}^{4} \psi_a^\dagger \mathrm{i}(\partial_t - v_a \partial_x)\psi_a + \mathcal{L}_{\text{int}}$$

- Four chiral fermion fields with a $\mathrm{U}(1) \times \mathrm{U}(1)'$ symmetry

$$\mathrm{U}(1) \times \mathrm{U}(1)' : \psi_a \to \mathrm{e}^{\mathrm{i}(q_a \theta + q'_a \theta')}\psi_a$$

fermion ψ_a	chirality v_a	charge (q_a, q'_a)
ψ_1	$+1$ (left)	$(3, 0)$
ψ_2	$+1$ (left)	$(4, 5)$
ψ_3	-1 (right)	$(5, 4)$
ψ_4	-1 (right)	$(0, 3)$



3-4-5-0 Model: SMG in (1+1)D

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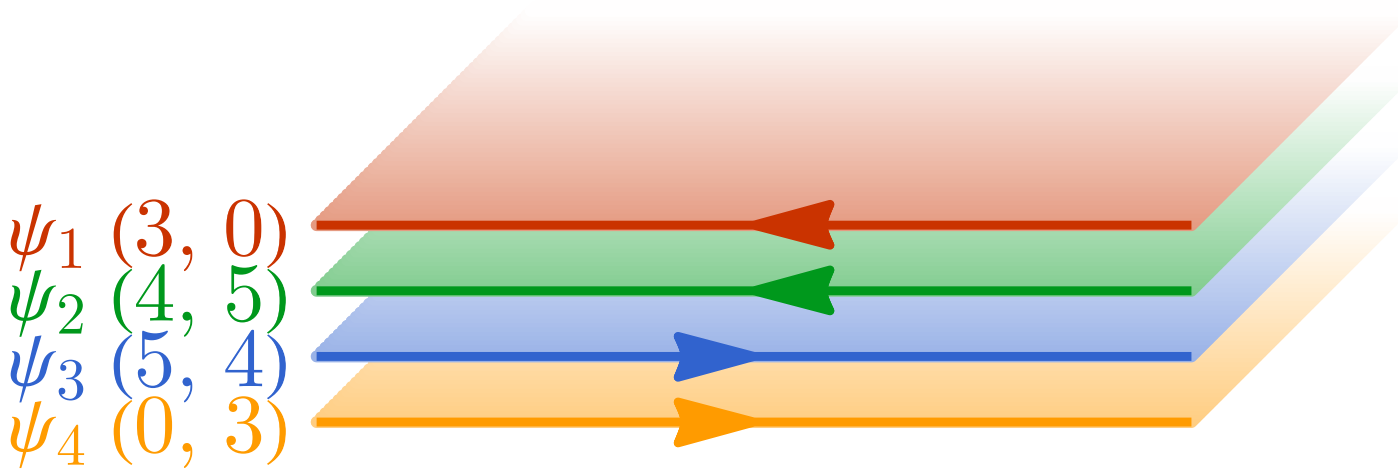


  - The system can be viewed as a (one-sided) boundary of a multi-layer (2+1)D quantum Hall insulator, each layer contributes to Hall conductances by (assuming $e^2/h = 1$)

$$\sigma_{\mathrm{H},a} = v_a q_a^2 \quad \sigma'_{\mathrm{H},a} = v_a q_a'^2 \qquad (a = 1, 2, 3, 4)$$

3-4-5-0 Model: SMG in (1+1)D

- Why the weird 3-4-5-0 charge assignment? - They are designed to enable SMG
  - $\mathrm{U}(1) \times \mathrm{U}(1)'$ must be anomaly free (no obstruction towards gapping), at least bulk Hall conductances must vanish

$$\sigma_{\mathrm{H}} = 3^2 + 4^2 - 5^2 - 0^2 = 0$$
$$\sigma'_{\mathrm{H}} = 0^2 + 5^2 - 4^2 - 3^2 = 0$$

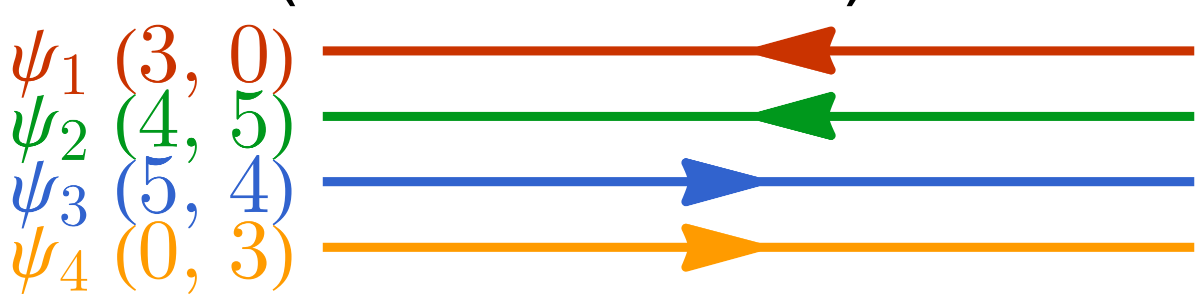
More generally, anomaly cancellation requires both self-anomaly and mixed-anomaly free

$$q_\alpha^\intercal K q_\beta = 0 \quad (\alpha, \beta = 1, 2)$$

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad q_1 := q = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix} \quad q_2 := q' = \begin{bmatrix} 0 \\ 5 \\ 4 \\ 3 \end{bmatrix}$$

3-4-5-0 Model: SMG in (1+1)D

- Why the weird 3-4-5-0 charge assignment? - They are designed to enable SMG
  - $\mathrm{U}(1) \times \mathrm{U}(1)'$ must be anomaly free (no obstruction towards gapping)
  - However, $\mathrm{U}(1) \times \mathrm{U}(1)'$ forbid any backscattering on the free-fermion level (no trivial mass)



Both Dirac masses $\psi_a^\dagger \psi_b$ and Majorana masses $\psi_a \psi_b$ are all charged under $\mathrm{U}(1) \times \mathrm{U}(1)'$

→ gapping is only possible by interaction effects

- What should be the correct interaction to drive SMG?

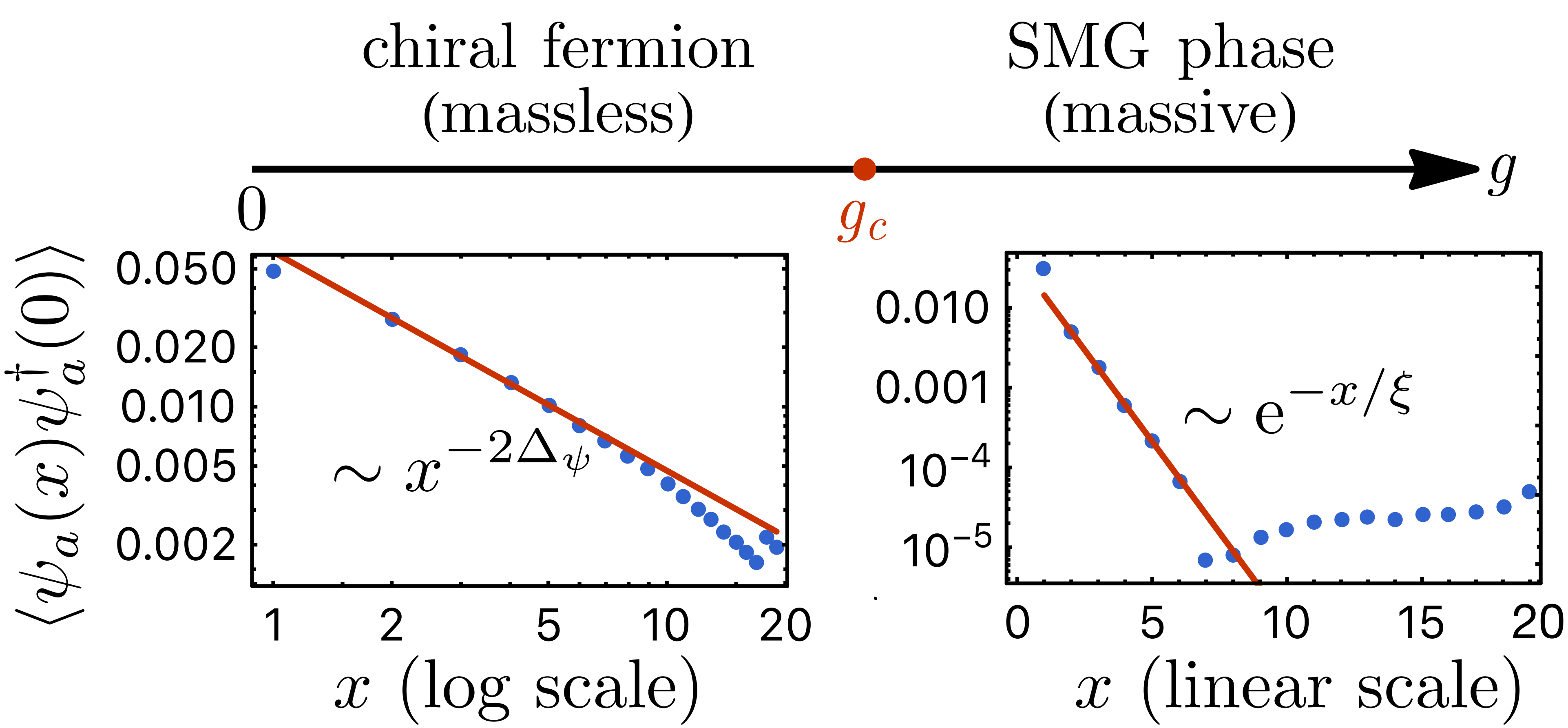
3-4-5-0 Model: SMG in (1+1)D

- The correct SMG interaction was proposed by Wang and Wen back in 2013 (and was recently verified by numerics)

$$\mathcal{L}_{\text{int}} = g_1(\psi_1\psi_2^\dagger\partial_x\psi_2^\dagger\psi_3\psi_4\partial_x\psi_4 + \text{h.c.}) + g_2(\psi_1\partial_x\psi_1\psi_2\psi_3^\dagger\partial_x\psi_3^\dagger\psi_3\psi_4 + \text{h.c.})$$

J Wang, XG Wen, 1307.7480, 1809.11171

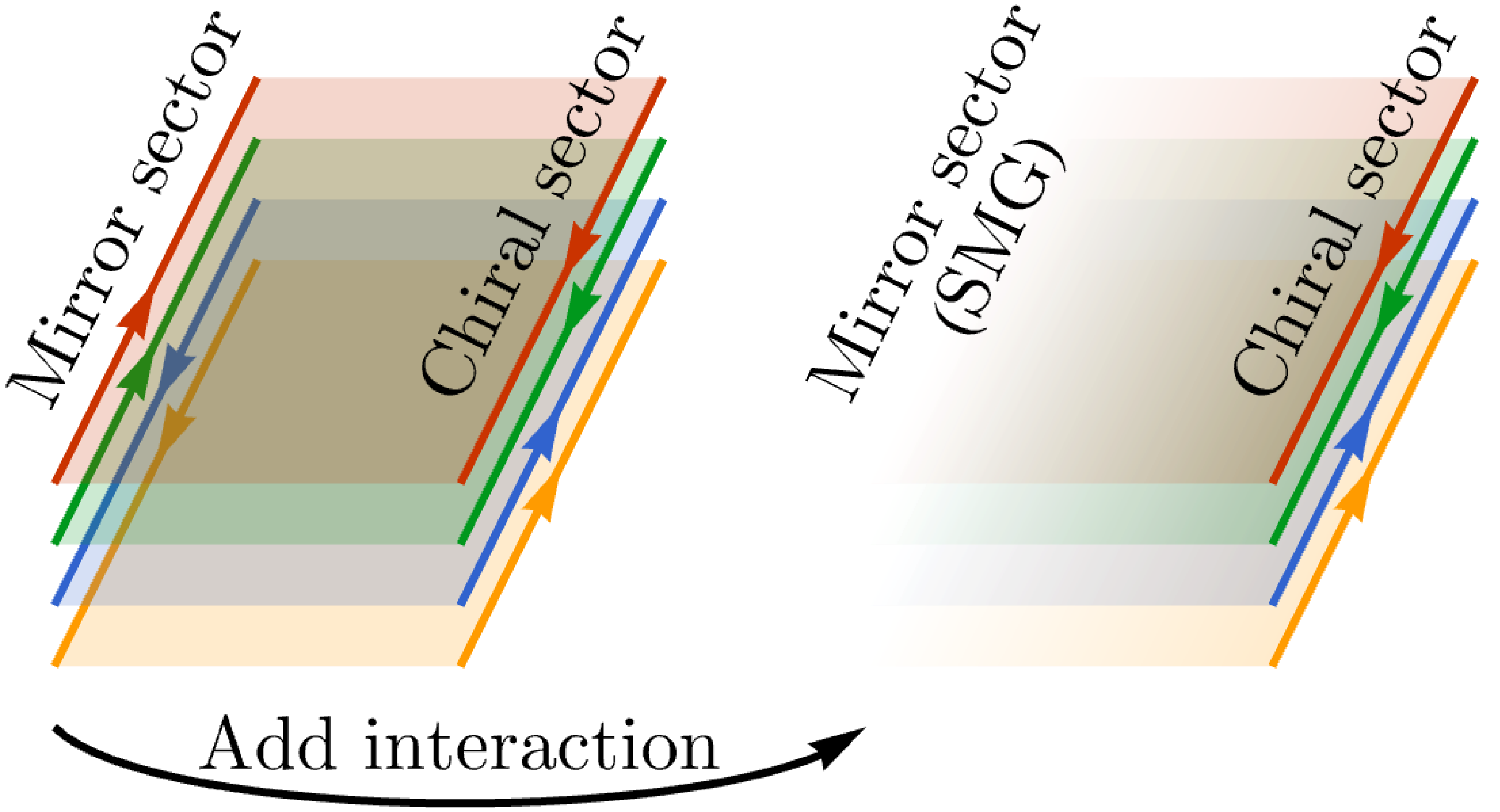
consider $g_1 = g_2 = g$, the phase diagram looks like this:



Zeng, Zhu, Wang, You, 2202.12355

3-4-5-0 Model: SMG in (1+1)D

- The significance of this result is that it demonstrates a new possibility to regularize chiral fermions on the lattice (at least in (1+1)D, hopefully, generalizable to (3+1)D)



- This is known as the mirror/domain wall fermion approach (which dates back to Eichten-Preskill 1986), but the correct gapping interaction was not known until Wang-Wen.

3-4-5-0 Model: SMG in (1+1)D

- Why the SMG interaction is so complicated? - In fact, Wang-Wen is already the most relevant interaction allowed by the gapping condition (i.e. anything simpler will not work)
- By bosonization $\psi_a \sim \mathrm{e}^{\mathrm{i}\varphi_a}$ ($a = 1, 2, 3, 4$), the fermion system can be equivalently described by a Littinger liquid theory

$$\mathcal{L} = \frac{1}{4\pi}(\partial_t \varphi^\intercal K \partial_x \varphi - \partial_x \varphi^\intercal V \partial_x \varphi) + \sum_{\alpha=1,2} g_\alpha \cos(l_\alpha^\intercal \varphi)$$

with

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad l_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix} \quad l_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

- Gapping condition: interaction operators must “braid” trivially

$$l_\alpha^\intercal K^{-1} l_\beta = 0 \quad (\alpha, \beta = 1, 2)$$

3-4-5-0 Model: SMG in (1+1)D

- Intuition: view the chiral fermions as the (1+1)D boundary of a (2+1)D $\mathrm{U}(1) \times \mathrm{U}(1)'$ gauge theory (enforcing symmetry on the boundary by gauging symmetry in bulk)
- A fully gapped boundary can only be consistently achieved by condensing the maximal set of bulk excitations $O_\alpha \sim \mathrm{e}^{\mathrm{i} l_\alpha^\intercal \varphi}$ that are self-boson and mutual-boson:

$$l_\alpha^\intercal K^{-1} l_\beta = 0 \quad (\alpha, \beta = 1, 2)$$

- Condensed operators $O_\alpha \sim \mathrm{e}^{\mathrm{i} l_\alpha^\intercal \varphi}$ must be neutral under the $\mathrm{U}(1) \times \mathrm{U}(1)'$ transformation (such that the interaction does not break the symmetry explicitly)

$$l_\alpha^\intercal q_\beta = 0 \quad (\alpha, \beta = 1, 2)$$

3-4-5-0 Model: SMG in (1+1)D

- Up to the freedom of basis choice, the solution is given as

$$l_\alpha = K q_\alpha$$

under which the anomaly cancellation, the symmetry requirement, and the gapping condition are all *consistent* with each other

$$q_\alpha^\intercal K q_\beta = l_\alpha^\intercal q_\beta = l_\alpha^\intercal K^{-1} l_\beta = 0$$

- Symmetry assignments dictate SMG interactions

$$q = \begin{bmatrix} 3 & 0 \\ 4 & 5 \\ 5 & 4 \\ 0 & 3 \end{bmatrix} \xrightarrow[K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}]{l_\alpha = K q_\alpha} l = \begin{bmatrix} 3 & 0 \\ 4 & 5 \\ -5 & -4 \\ 0 & -3 \end{bmatrix}$$

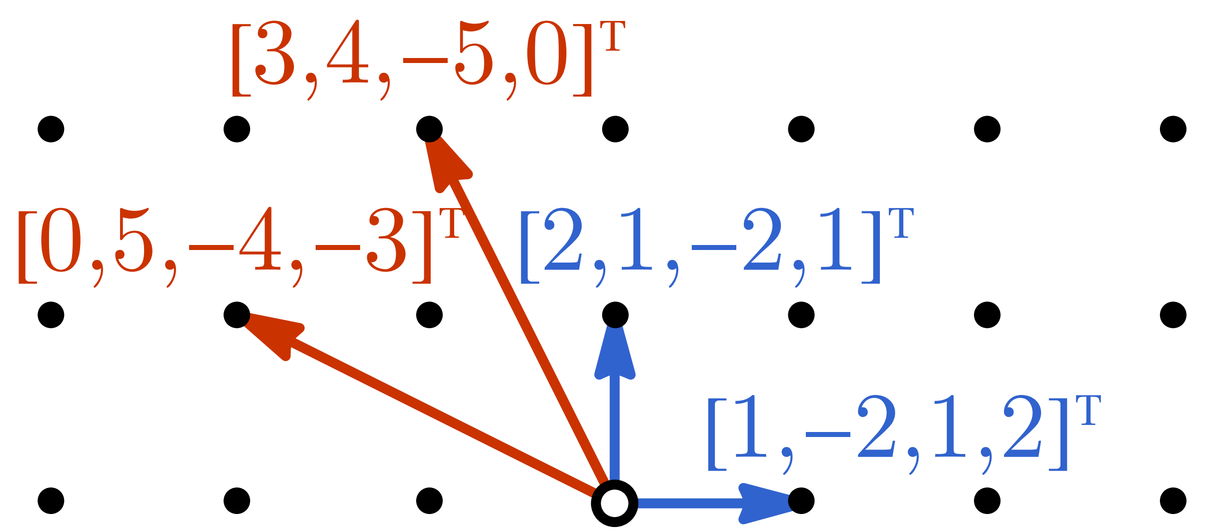
Charge assignments

Condensible operator basis

3-4-5-0 Model: SMG in (1+1)D

- The lattice of condensible operators (condensible algebra)

$$\{O_l \sim e^{\mathrm{i}l^\intercal\varphi} | l \in \mathrm{span}(l_1, l_2) \cap \mathbb{Z}^4\}$$



- Operator scaling dimension $\Delta_l = \frac{1}{2} l^\intercal l$ (at the free-fermion fixed point) → shorter l vector = more relevant O_l operator

$$O_{[1,-2,1,2]^\intercal} = \psi_1 \psi_2^\dagger \partial_x \psi_2^\dagger \psi_3 \psi_4 \partial_x \psi_4$$

$$O_{[2,1,-2,1]^\intercal} = \psi_1 \partial_x \psi_1 \psi_2 \psi_3^\dagger \partial_x \psi_3^\dagger \psi_3 \psi_4$$

3-4-5-0 Model: SMG in (1+1)D

- The SMG interaction is designed to drive the condensation of these (maximally) condensible operators

$$\mathcal{L}_{\text{int}} = g_1 O_{l_1} + g_2 O_{l_2} + \text{h.c.}$$

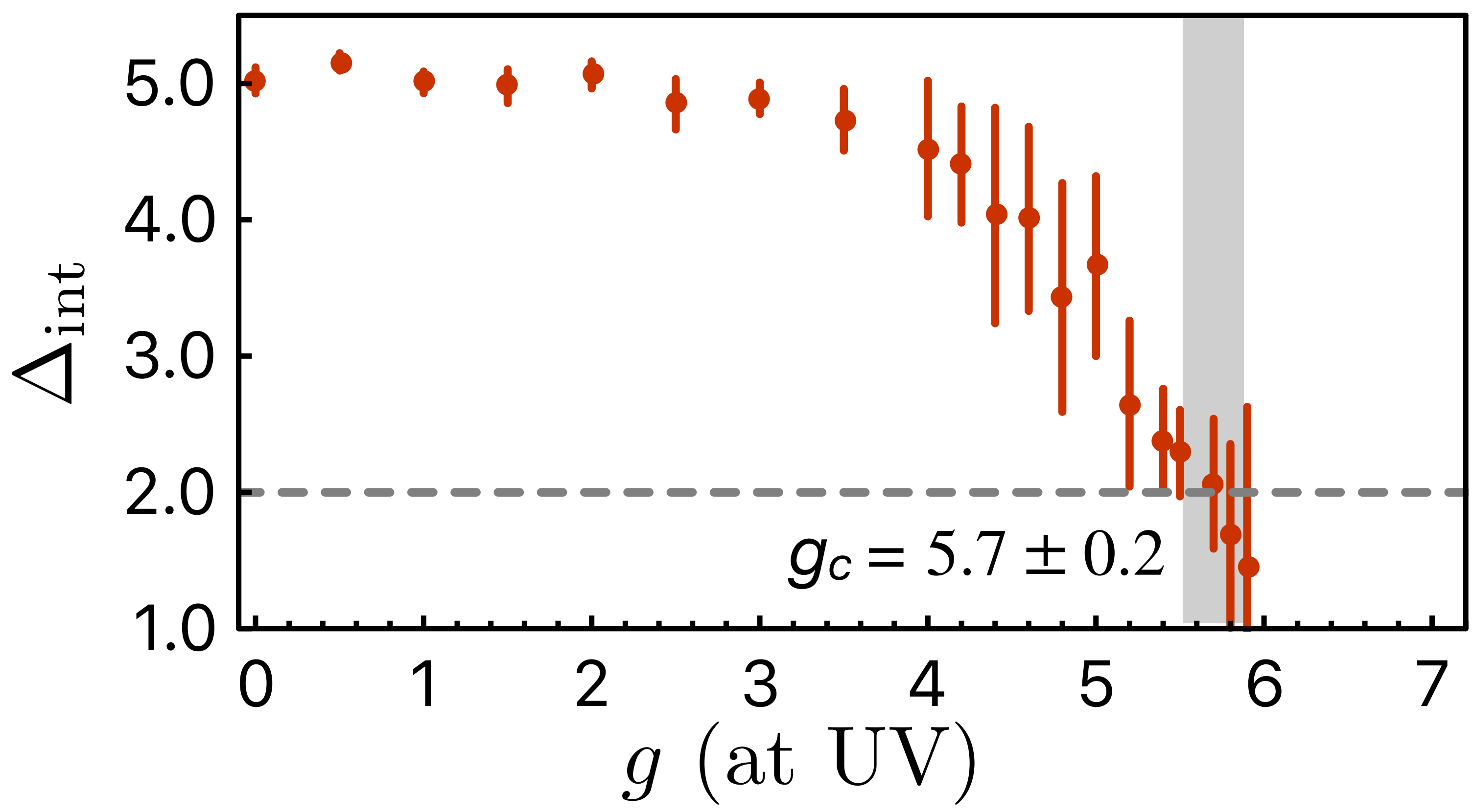
- Even though O_{l_α} have been chosen to be the most relevant operators in the condensible algebra, their scaling dimension at the free-fermion fixed point is still pretty high

$$\Delta_{\text{int}} = \frac{1}{2} l_\alpha^\intercal l_\alpha = 5 > 2$$

  - High-energy physics: adding these irrelevant operators makes the field theory unrenormalizable …
  - Condensed matter physics: adding these irrelevant operators opens up new opportunities toward adjacent phases of matters!

3-4-5-0 Model: SMG in (1+1)D

- If the interaction is turned on perturbatively, it will flow to 0.
- But there can be non-perturbative effects when the coupling is strong enough



- The interaction renormalizes the Luttinger parameter(s), which in turn reduces its own scaling dimension

3-4-5-0 Model: SMG in (1+1)D

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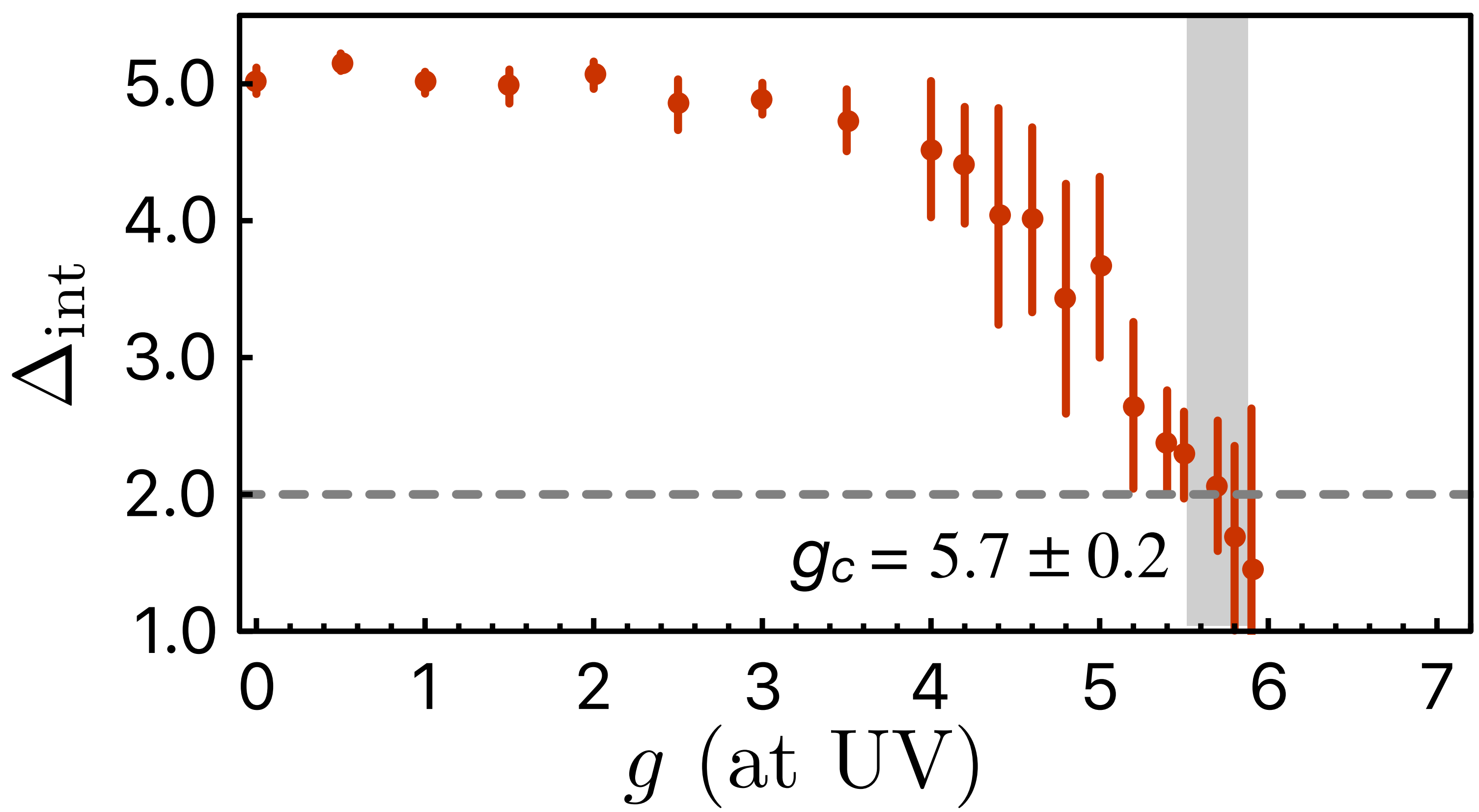


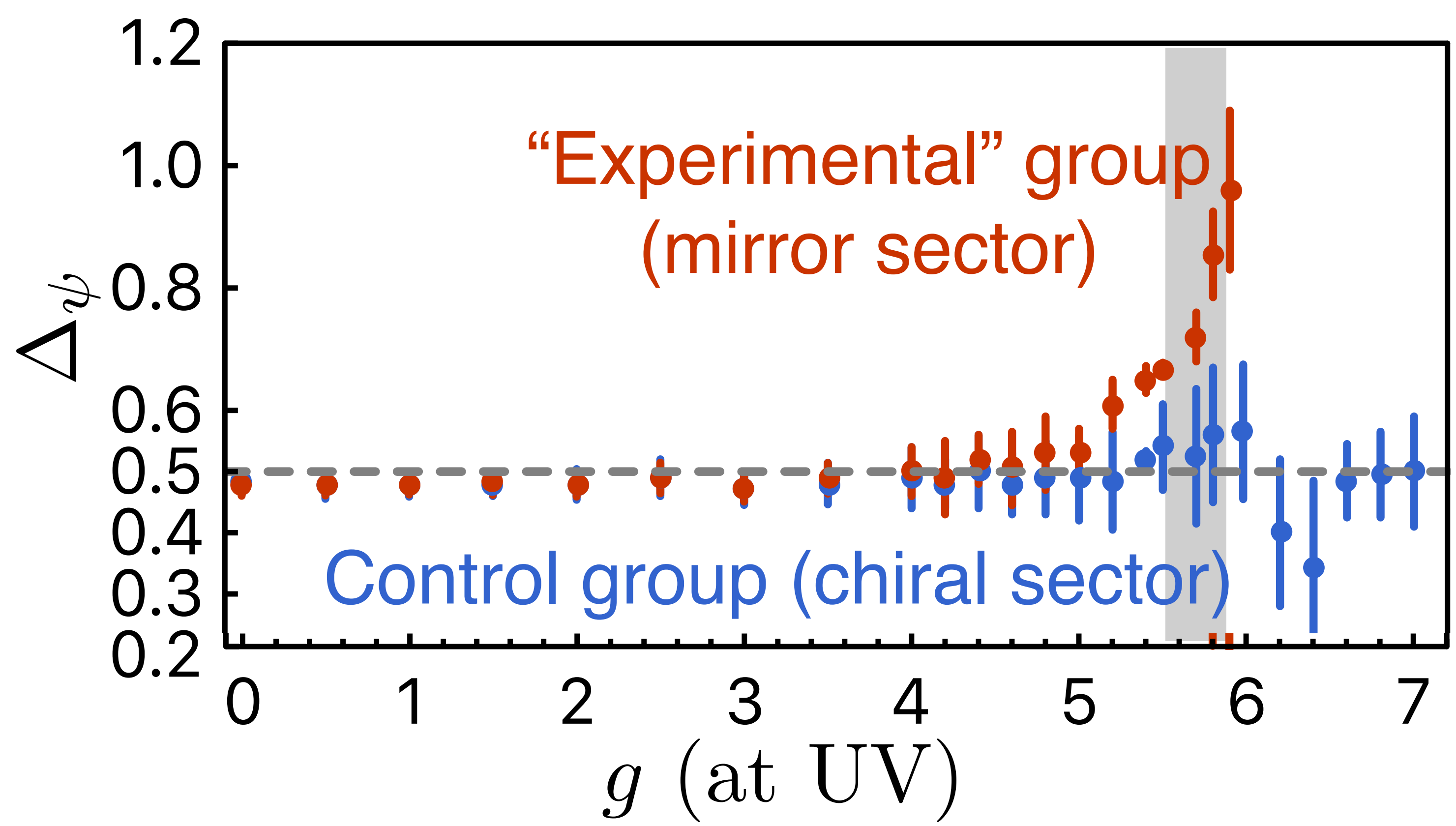
- Transition happens when $\Delta_{\text{int}} = 2$ (where the interaction becomes marginal) → leading to a BKT transition

3-4-5-0 Model: SMG in (1+1)D

- If the interaction is turned on perturbatively, it will flow to 0.
- But there can be non-perturbative effects when the coupling is strong enough
  - Beyond this point (when $g > g_c$), the interaction is relevant and flows strong under RG → driving all condensible operators to condense
  - The remaining operators that braid non-trivially with the condensed operators will all be gapped, e.g. the fermion operator → mass (gap) generation for fermions

3-4-5-0 Model: SMG in (1+1)D

- At the SMG critical point, the fermion operator must have a higher scaling dimension, as the fermion correlation is decaying faster in the SMG phase compared to the chiral fermion phase.



- The increasing fermion scaling dimension is a precursor of fractionalization (which can happen in higher dimensions).

Time to Break.

Bilayer Honeycomb Model: SMG in (2+1)D

- How can we extend our understanding of SMG to higher dimensions?
  - (0+1)D: interacting fermions are exact solvable
  - (1+1)D: interacting fermions can be bosonized
  - (2+1)D and above: the above techniques fail …
- New idea: Fermion fractionalization
  - A unified framework to understand the SMG critical point in higher dimensions
  - Hypothesis: physical fermions fractionalizes into deconfined partons at and only at the SMG critical point → a fermionic version of the deconfined quantum critical point (fDQCP)

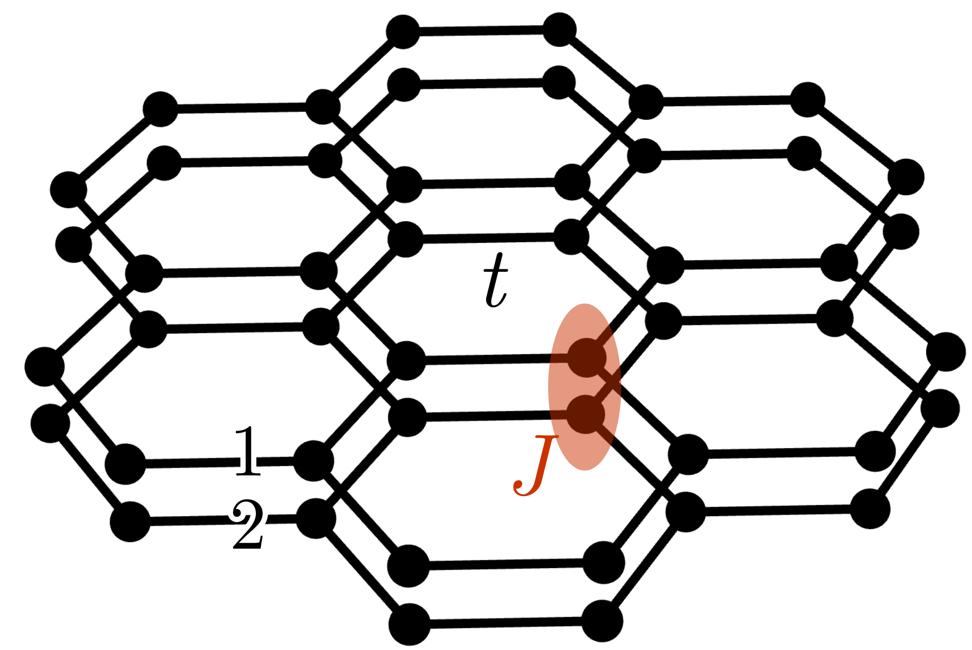
YZ You, YC He, C Xu, A Vishwanath, 1705.09313; 1711.00863

Bilayer Honeycomb Model: SMG in (2+1)D

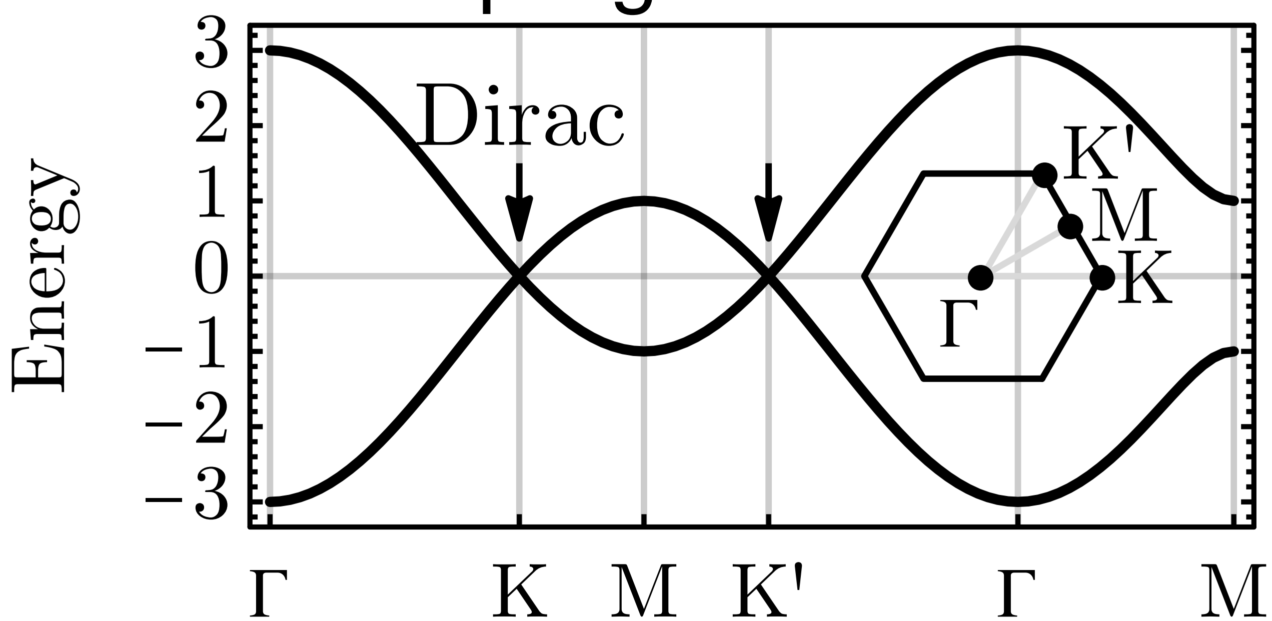
- Bilayer honeycomb model K Slagle, YZ You, C Xu, 1409.7401

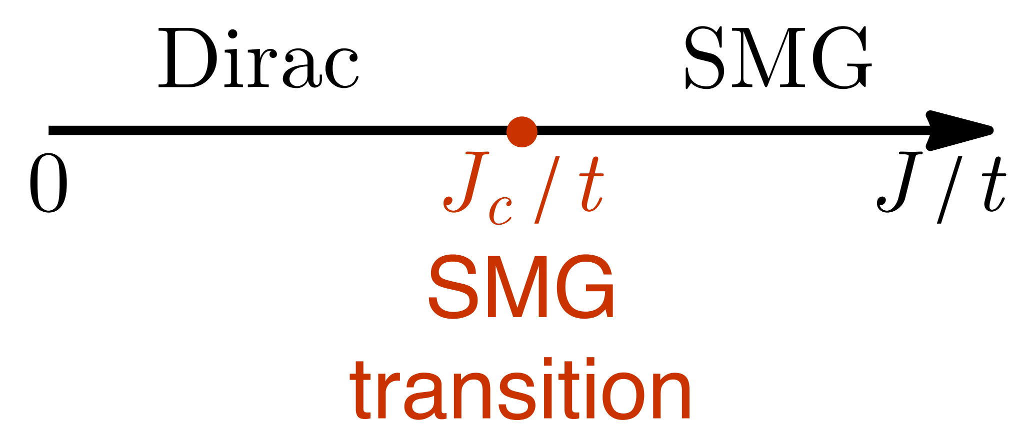
$$H = -t \sum_{\langle ij \rangle, l, \sigma} (c^\dagger_{il\sigma} c_{jl\sigma} + \text{h.c.}) + J \sum_i \boldsymbol{S}_{i1} \cdot \boldsymbol{S}_{i2},$$

- Every site: $l = 1, 2; \sigma = \uparrow, \downarrow$
  - $c_{il\sigma}$: electron operator
  - $\boldsymbol{S}_{il} = \frac{1}{2} c^\dagger_{il} \boldsymbol{\sigma} c_{il}$: spin operator



- Weak coupling: Dirac semi-metal





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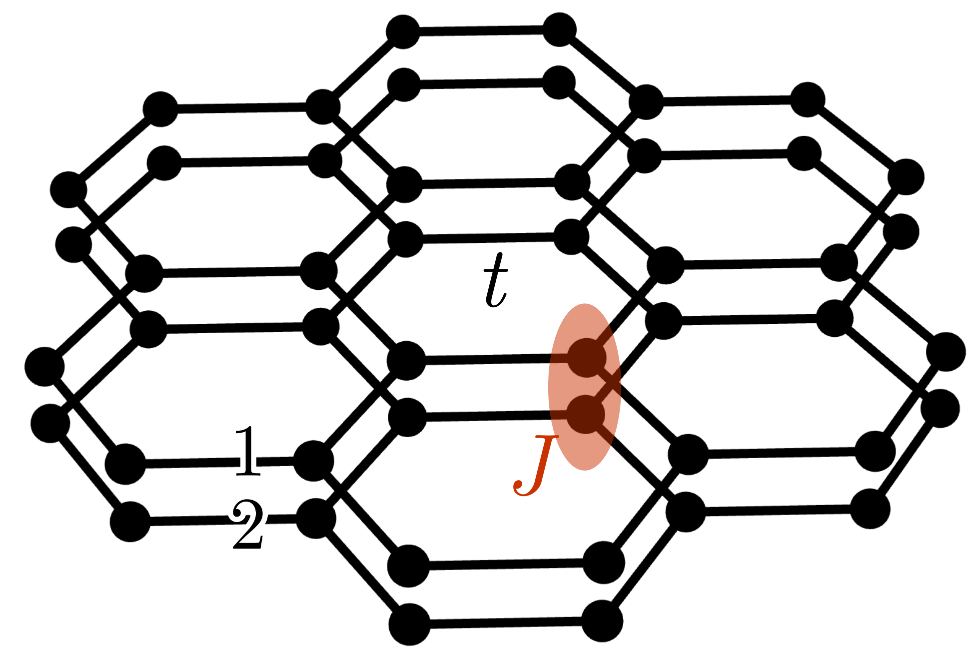
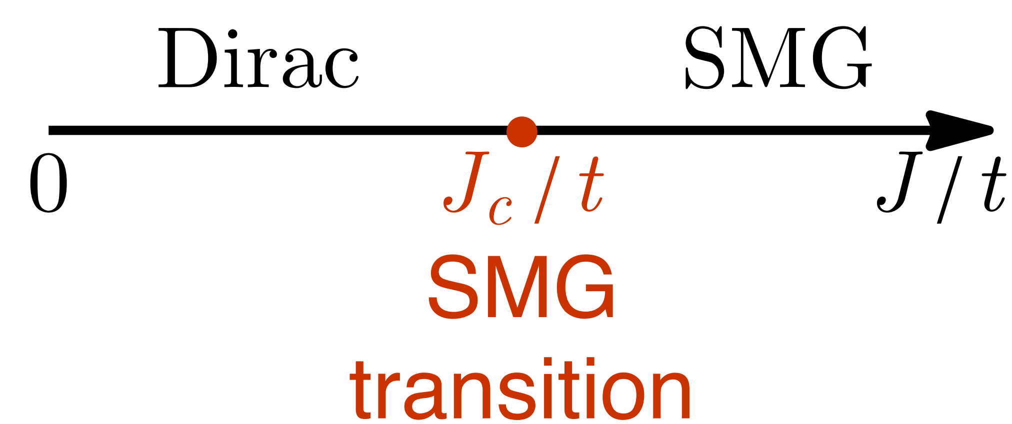
- Every site: $l = 1, 2; \sigma = \uparrow, \downarrow$
  - $c_{il\sigma}$: electron operator
  - $\boldsymbol{S}_{il} = \frac{1}{2} c^\dagger_{il} \boldsymbol{\sigma} c_{il}$: spin operator



- Strong coupling: SMG insulator
  Ground state = product of inter-layer spin singlets

$$\bigotimes_i \frac{1}{\sqrt{2}} (|\uparrow_1 \downarrow_2\rangle - |\downarrow_1 \uparrow_2\rangle)_i$$

with a gap to all excitations.



Bilayer Honeycomb Model: SMG in (2+1)D

- Bilayer honeycomb model   K Slagle, YZ You, C Xu, 1409.7401

$$H = -t \sum_{\langle ij \rangle, l, \sigma} (c_{il\sigma}^{\dagger} c_{jl\sigma} + \text{h.c.}) + J \sum_{i} \boldsymbol{S}_{i1} \cdot \boldsymbol{S}_{i2},$$

- The model has (at least) an $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \mathrm{SU}(2) \times \mathbb{Z}_2^{\mathcal{S}}$ internal symmetry and the honeycomb lattice symmetry
  - $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$: charge conservation in separate layers

$$\mathrm{U}(1)_l : c_{il} \to \mathrm{e}^{\mathrm{i}\theta_l} c_{il}$$

  - $\mathrm{SU}(2)$: spin conservation (across layers)

$$\mathrm{SU}(2) : c_{il} \to \mathrm{e}^{\frac{\mathrm{i}}{2} \boldsymbol{\theta} \cdot \boldsymbol{\sigma}} c_{il}$$

  - $\mathbb{Z}_2^{\mathcal{S}}$: sublattice charge conjugation symmetry (anti-unitary)

$$\mathbb{Z}_2^{\mathcal{S}} : c_{il} \to (-1)^i c_{il}^{\dagger}, \mathrm{i} \to -\mathrm{i}$$

  - Lattice symmetry: translations, rotations, reflections …

Bilayer Honeycomb Model: SMG in (2+1)D

- Bilayer honeycomb model K Slagle, YZ You, C Xu, 1409.7401

$$H = -t \sum_{\langle ij \rangle, l, \sigma} (c^\dagger_{il\sigma} c_{jl\sigma} + \text{h.c.}) + J \sum_i \boldsymbol{S}_{i1} \cdot \boldsymbol{S}_{i2},$$

- The model has (at least) an $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \mathrm{SU}(2) \times \mathbb{Z}_2^{\mathcal{S}}$ internal symmetry and the honeycomb lattice symmetry
- With these symmetries, it is impossible to gap out the Dirac fermions by any fermion bilinear terms.
- For example, one may attempt to create a bilinear mass gap by introducing a staggered interlayer hopping term

$$H \to H + H_m \qquad H_m = \sum_i m_i c^\dagger_{i1} c_{i2} + \text{h.c.}$$ (Attempt only!)

$$m_i = (-1)^i m = \pm m \text{ for } i \in A/B$$

Bilayer Honeycomb Model: SMG in (2+1)D

- Bilayer honeycomb model K Slagle, YZ You, C Xu, 1409.7401

$$H = -t \sum_{\langle ij \rangle, l, \sigma} (c_{il\sigma}^\dagger c_{jl\sigma} + \mathrm{h.c.}) + J \sum_i \boldsymbol{S}_{i1} \cdot \boldsymbol{S}_{i2},$$

- The model has (at least) an $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2 \times \mathrm{SU}(2) \times \mathbb{Z}_2^{\mathcal{S}}$ internal symmetry and the honeycomb lattice symmetry
- One attempt to open a bilinear gap:

$$H_m = \sum_i m_i c_{i1}^\dagger c_{i2} + \mathrm{h.c.}$$ (Attempt only!)

- However, this will break the inter-layer $\mathrm{U}(1)_-$ and the $\mathbb{Z}_2^{\mathcal{S}}$ symmetries, since

$$\mathrm{U}(1)_- : c_{i1} \to \mathrm{e}^{\mathrm{i}\theta_-} c_{i1}, c_{i2} \to \mathrm{e}^{-\mathrm{i}\theta_-} c_{i2}, m_i \to \mathrm{e}^{2\mathrm{i}\theta_-} m_i$$

$$\mathbb{Z}_2^{\mathcal{S}} : c_{il} \to (-1)^i c_{il}^\dagger, m_i \to -m_i$$

Bilayer Honeycomb Model: SMG in (2+1)D

- What we learn from condensed matter physics: if you can not open a gap for physical fermions, you can try it on fermionic partons (gauged fermions)!
  - Example: quantum spin liquid - fail to open a superconducting gap in Mott insulators, open it for fermionic spinons by spin fractionalization.
  - Analogy: SMG - fail to open a fermion bilinear gap in SMG insulators, open it for fermionic partons by fermion fractionalization.
- Consider writing the electron operator c_{il} as the product of a boson operator b_{il} and a fermion operator f_{il} on every site and layer (electron spin will be assigned to f_{il})

$$c_{il} = \begin{bmatrix} c_{il\uparrow} \\ c_{il\downarrow} \end{bmatrix} = b_{il} \begin{bmatrix} f_{il\uparrow} \\ f_{il\downarrow} \end{bmatrix} = b_{il} f_{il}$$

Bilayer Honeycomb Model: SMG in (2+1)D

- As if the electron c_{il} were not a fundamental particle but a composite particle

$$c_{il} = b_{il} f_{il}$$

made of a bosonic parton b_{il} and a fermionic parton f_{il}.

- This rewriting is called fermion fractionalization.
- It comes with a price (or a gift?): the emergent gauge structure - as the partons are now *redundant* descriptions of the original physical electron that the following transformation is *unphysical* (i.e. no physical effect)

$$b_{il} \to \mathrm{e}^{-\mathrm{i}\theta_{il}} b_{il}$$

$$f_{il} \to \mathrm{e}^{\mathrm{i}\theta_{il}} f_{il}$$

- The emergent gauge group is $\tilde{\mathrm{U}}(1)_1 \times \tilde{\mathrm{U}}(1)_2$ (add a tilde to avoid confusion with the $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ symmetry)

Bilayer Honeycomb Model: SMG in (2+1)D

- Charge assignments (on every site)

	$\tilde{\mathrm{U}}(1)_1$	$\tilde{\mathrm{U}}(1)_2$	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{SU}(2)$
c_{i1}	0	0	1	0	$\mathbf{2}$
c_{i2}	0	0	0	1	$\mathbf{2}$
m_i	0	0	1	-1	$\mathbf{1}$
b_{i1}	-1	0	1	0	$\mathbf{1}$
b_{i2}	0	-1	0	1	$\mathbf{1}$
f_{i1}	1	0	0	0	$\mathbf{2}$
f_{i2}	0	1	0	0	$\mathbf{2}$
M_i	1	-1	0	0	$\mathbf{1}$

- Now the fermionic parton bilinear mass can be condensed without breaking symmetry, but only to drive gauge Higgsing

$$H_M = \sum_i M_i f_{i1}^\dagger f_{i2} + \mathrm{h.c.}$$

Bilayer Honeycomb Model: SMG in (2+1)D

- Effective field theory description

$$\mathcal{L} = \sum_{l=1,2} \Big(|(\partial - \mathrm{i}(A_l - a_l))b_l|^2 + r|b_l|^2 + u|b_l|^4 + \bar{f}_l\, \gamma \cdot (\partial - \mathrm{i}a_l) f_l \Big)$$

  - b_l: single-component (per layer) scalar field
  - f_l: four-component (per layer) spinor field (Dirac fermion)
    $f_l = \begin{bmatrix} f_{lK\uparrow} & f_{lK\downarrow} & f_{lK'\uparrow} & f_{lK'\downarrow} \end{bmatrix}^{\mathsf{T}}$ (2 valleys x 2 spins)
  - a_l: dynamical $\tilde{\mathrm{U}}(1)_l$ 1-form gauge field
  - A_l: background $\mathrm{U}(1)_l$ 1-form gauge field, serving as symmetry probe field
- The SMG tuning parameter is the bosonic parton mass r

Bilayer Honeycomb Model: SMG in (2+1)D

- Effective field theory description

$$\mathcal{L} = \sum_{l=1,2} \Big(|(\partial - \mathrm{i}(A_l - a_l))b_l|^2 + r|b_l|^2 + u|b_l|^4 + \bar{f}_l \, \gamma \cdot (\partial - \mathrm{i}a_l) f_l \Big)$$

- The SMG tuning parameter is the bosonic parton mass r
  - $r < 0$: bosonic partons b_l condense, pinning gauge fields a_l to background fields A_l through the Higgs mechanism, such that fermionic partons f_l regain the $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ symmetry and become physical fermions
    → Dirac semi-metal phase

$$\mathcal{L} = \sum_{l=1,2} \bar{c}_l \, \gamma \cdot (\partial - \mathrm{i}A_l) c_l$$

Bilayer Honeycomb Model: SMG in (2+1)D

- Effective field theory description

$$\mathcal{L} = \sum_{l=1,2} \Big(|(\partial - \mathrm{i}(A_l - a_l))b_l|^2 + r|b_l|^2 + u|b_l|^4 + \bar{f}_l \, \gamma \cdot (\partial - \mathrm{i}a_l) f_l \Big)$$

- The SMG tuning parameter is the bosonic parton mass r
  - $r > 0$: bosonic partons b_l gapped and decoupled, fermionic partons f_l spontaneous develop parton-Higgs mass $\bar{f}_1 f_2$ acquiring the gap while Higgsing gauge fields a_l to the diagonal $\tilde{\mathrm{U}}(1)_+$ which confines automatically by monopole proliferation (Polyakov)
    → SMG insulator phase

Bilayer Honeycomb Model: SMG in (2+1)D

- Effective field theory description

$$\mathcal{L} = \sum_{l=1,2} \Big(|(\partial - \mathrm{i}(A_l - a_l))b_l|^2 + r|b_l|^2 + u|b_l|^4 \\ + \bar{f}_l \, \gamma \cdot (\partial - \mathrm{i}a_l) f_l \Big)$$

- The SMG tuning parameter is the bosonic parton mass r
  - The SMG transition happens at $r = 0$.
  - This is a deconfined quantum critical point (DQCP) because away from the transition (either $r > 0$ or $r < 0$), gauge fields are Higgsed / confined. Partons are deconfined at and only at the SMG critical point.
  - This is a fermionic DQCP in the sense that fermions (other than bosonic order parameters) are fractionalizing here.

Bilayer Honeycomb Model: SMG in (2+1)D

- At the SMG critical point

$$\mathcal{L} = \sum_{l=1,2} |(\partial - \mathrm{i}(A_l - a_l))b_l|^2 + u|b_l|^4 + \bar{f}_l \gamma \cdot (\partial - \mathrm{i}a_l) f_l$$

- Two layers are decoupled. Each layer: a QED_3 theory with $N_b = 1$ bosons (scalars) and $N_f = 4$ fermions (spinors).
- Prediction: Large-N_b, N_f estimation of the scaling dimension for physical fermions $c_l = b_l f_l$ gives

$$\Delta_c \simeq 1.3 > 1$$

R Kaul, S Sachdev, 0801.0723

i.e. electron two-point correlation should decay faster at the SMG critical point with a larger power compared to the free Dirac fermion. (This has not been tested by numerics yet …)

Bilayer Honeycomb Model: SMG in (2+1)D

- Deep in the SMG phase, the gauge confinement is so strong that it essentially enforces gauge projection on each site → this provides a local picture for SMG
- Starting from the parton-Higgs mass

$$H_M = \sum_i M_i f_{i1}^\dagger f_{i2} + \text{h.c.}$$

  - Sites are decoupled. Each site has the ground state

$$\begin{aligned}|\Psi_i\rangle &= \prod_{\sigma=\uparrow,\downarrow} \frac{1}{\sqrt{2}}\Big(f_{i1\sigma}^\dagger - \frac{M_i^*}{|M_i|} f_{i2\sigma}^\dagger\Big)|\text{vac}\rangle \\ &\propto -\Big(\frac{M_i}{|M_i|} f_{i1\uparrow}^\dagger f_{i1\downarrow}^\dagger + \frac{M_i^*}{|M_i|} f_{i2\uparrow}^\dagger f_{i2\downarrow}^\dagger\Big)|\text{vac}\rangle \\ &\quad + \big(f_{i1\uparrow}^\dagger f_{i2\downarrow}^\dagger - f_{i1\downarrow}^\dagger f_{i2\uparrow}^\dagger\big)|\text{vac}\rangle\end{aligned}$$

Bilayer Honeycomb Model: SMG in (2+1)D

- Because the parton-Higgs mass M_i is not gauge-neutral

$$\tilde{\mathrm{U}}(1)_- : M_i \to \mathrm{e}^{2\mathrm{i}\tilde{\theta}_{-,i}} M_i$$

terms that depend on the phase of M_i can not survive the gauge projection

$$|\Psi_i\rangle \propto -\Big(\frac{M_i}{|M_i|} f^\dagger_{i1\uparrow} f^\dagger_{i1\downarrow} + \frac{M_i^*}{|M_i|} f^\dagger_{i2\uparrow} f^\dagger_{i2\downarrow}\Big)|\mathrm{vac}\rangle + \big(f^\dagger_{i1\uparrow} f^\dagger_{i2\downarrow} - f^\dagger_{i1\downarrow} f^\dagger_{i2\uparrow}\big)|\mathrm{vac}\rangle$$

Confinement

$$\begin{aligned} P_i|\Psi_i\rangle &= \tfrac{1}{\sqrt{2}}(f^\dagger_{i1\uparrow} f^\dagger_{i2\downarrow} - f^\dagger_{i1\downarrow} f^\dagger_{i2\uparrow})|\mathrm{vac}\rangle \\ &= \tfrac{1}{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \end{aligned}$$

- Reproducing the exact ground state in the $J \to \infty$ limit.

Fermion Green's Function Zero

- What did we learn from the above calculation?
  "Symmetric" mass (in SMG) ~ parton bilinear mass ~ physical bilinear mass disordered by fluctuations
- This has an important implication for the fermion Green's function (two-point correlation)

$$\mathcal{G}(x) := \langle\bar{\psi}(0)\psi(x)\rangle = \frac{1}{Z}\int\mathcal{D}[\psi]\bar{\psi}(0)\psi(x)\mathrm{e}^{\mathrm{i}S[\psi]}$$

  - For free-fermions,

$$S[\psi] = \int \mathrm{d}^d x\ \bar{\psi}(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi$$

    the answer is (in momentum space)

$$\mathcal{G}(k) = \frac{\gamma^\mu k_\mu + m}{k^\mu k_\mu - |m|^2}$$

Fermion Green's Function Zero

- This Green's function has the following features:

$$\mathcal{G}(k) = \frac{\gamma^\mu k_\mu + m}{k^\mu k_\mu - |m|^2}$$

Poles along $k^\mu k_\mu - |m|^2 = 0$

Dispersion: $\epsilon_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + |m|^2}$

  - Rest mass: the energy gap to fermion excitations

$$m_{\text{rest}} = \min_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} = |m|$$

  - Inertial mass: the inverse curvature of fermion dispersion

$$m_{\text{iner}} = \lim_{\boldsymbol{k}\to 0} (\partial^2_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}})^{-1} = |m|$$

  - Bilinear mass condensation

$$\langle \bar{\psi}\psi \rangle \sim m f(|m|) \neq 0$$

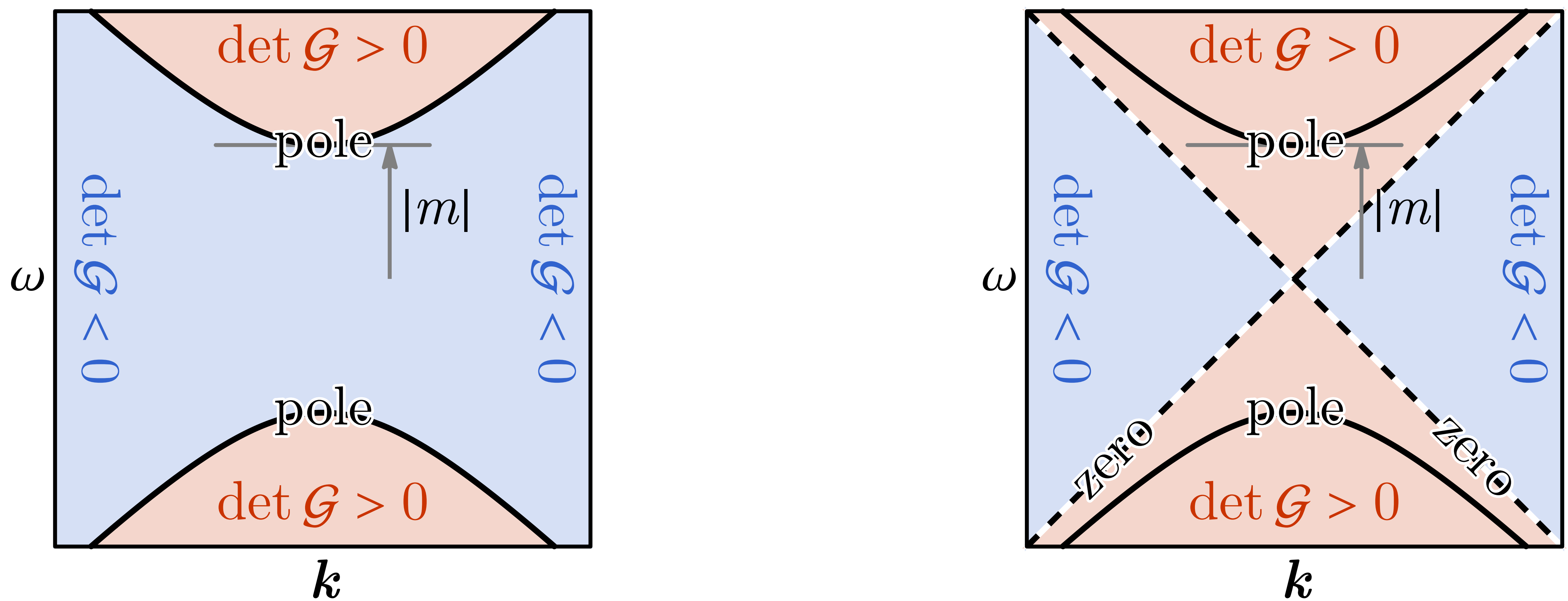
- However, different "masses" may not always be equivalent.

Fermion Green's Function Zero

- SMG = Disordering the bilinear mass without tuning off its amplitude

$$\mathcal{G}(k) = \frac{\gamma^\mu k_\mu + m}{k^\mu k_\mu - |m|^2} \xrightarrow[\overline{|m|^2} \neq 0]{\overline{m} = 0} \mathcal{G}(k) = \frac{\gamma^\mu k_\mu}{k^\mu k_\mu - |m|^2}$$

Free-massive fermion | SMG fermion



Fermion Green's Function Zero

- Fermion Green's function (deep) in the SMG phase

$$\mathcal{G}(k) = \frac{\gamma^\mu k_\mu}{k^\mu k_\mu - |m|^2}$$

  - Poles along $k^\mu k_\mu - |m|^2 = 0$ → quasi-particle excitations are still well-defined above the gap with finite rest mass and inertial mass

$$m_{\text{rest}} = m_{\text{iner}} = |m|$$

  - Zeros along $k^\mu k_\mu = 0$ → no bilinear mass condensation

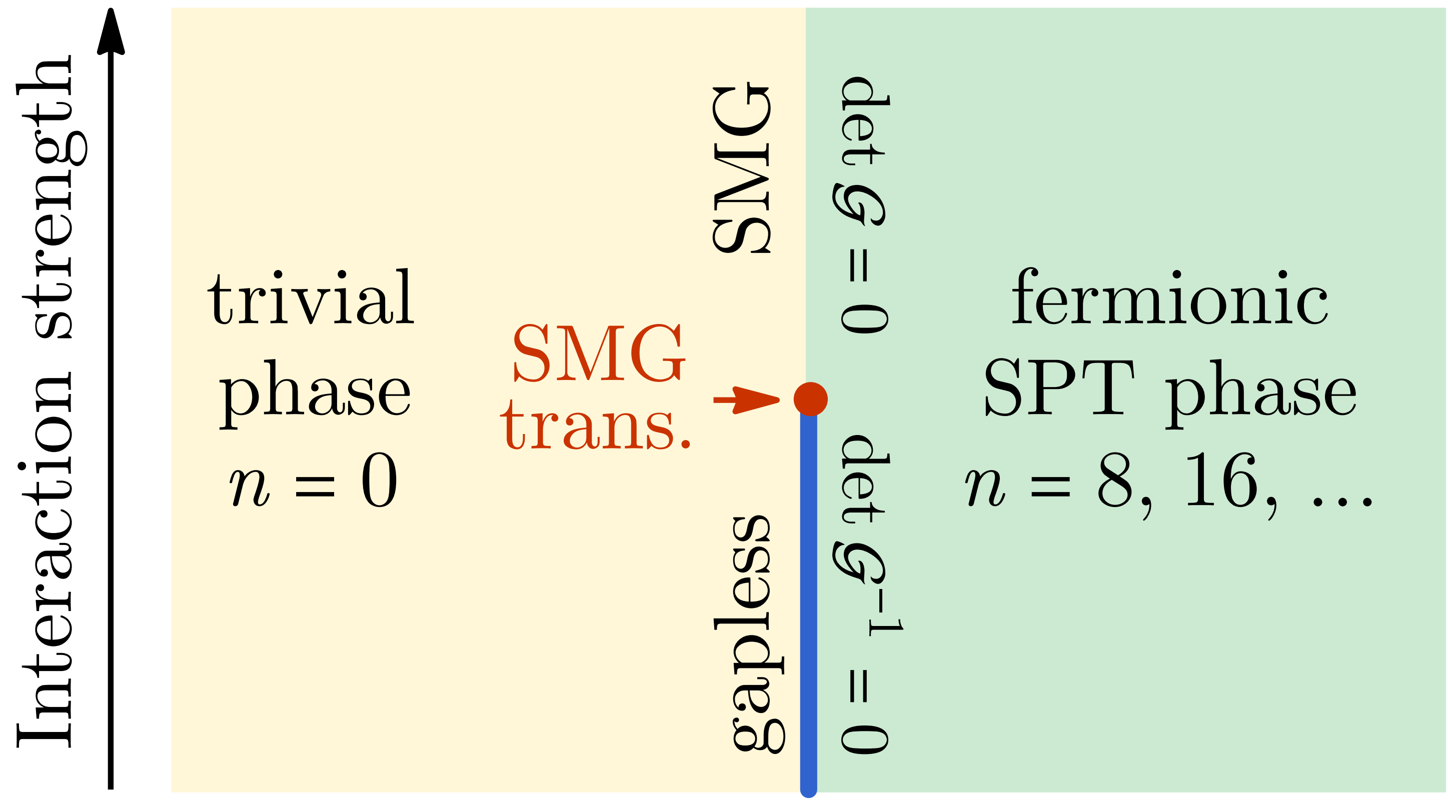
$$\langle \bar{\psi}\psi \rangle = \int \mathrm{d}^d k \, \mathcal{G}(k) = 0$$

    as $\mathcal{G}(k)$ is odd in k_μ (as required by symmetry)

- $\det \mathcal{G}(\omega = 0) = 0$ is a non-perturbative robust feature of SMG!

Fermion Green's Function Zero

- To see that $\det \mathcal{G} = 0$ must happen in the SMG phase:

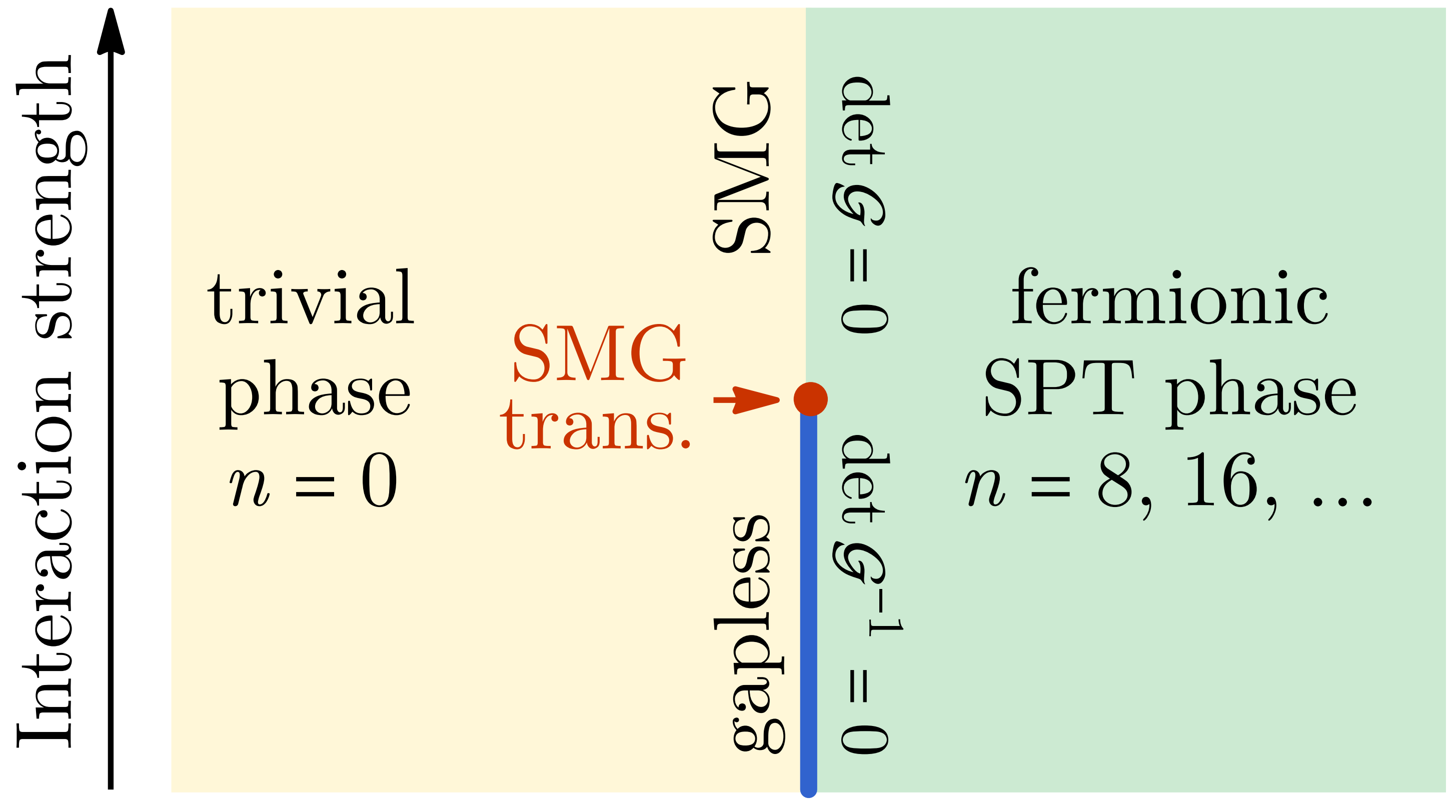


- Consider gapless fermions on the boundary between trivial and topological insulators
- Apply SMG interaction on the boundary with a gradient in the vertical direction (along the boundary)
- The gapless fermions will end at the SMG transition.

YZ You, Z Wang, J Oon, C Xu, 1403.4938

Fermion Green's Function Zero

- To see that $\det \mathcal{G} = 0$ must happen in the SMG phase:



- Now we compute the (free-fermion) topological index on both sides in the bulk (where the bulk is non-interacting)

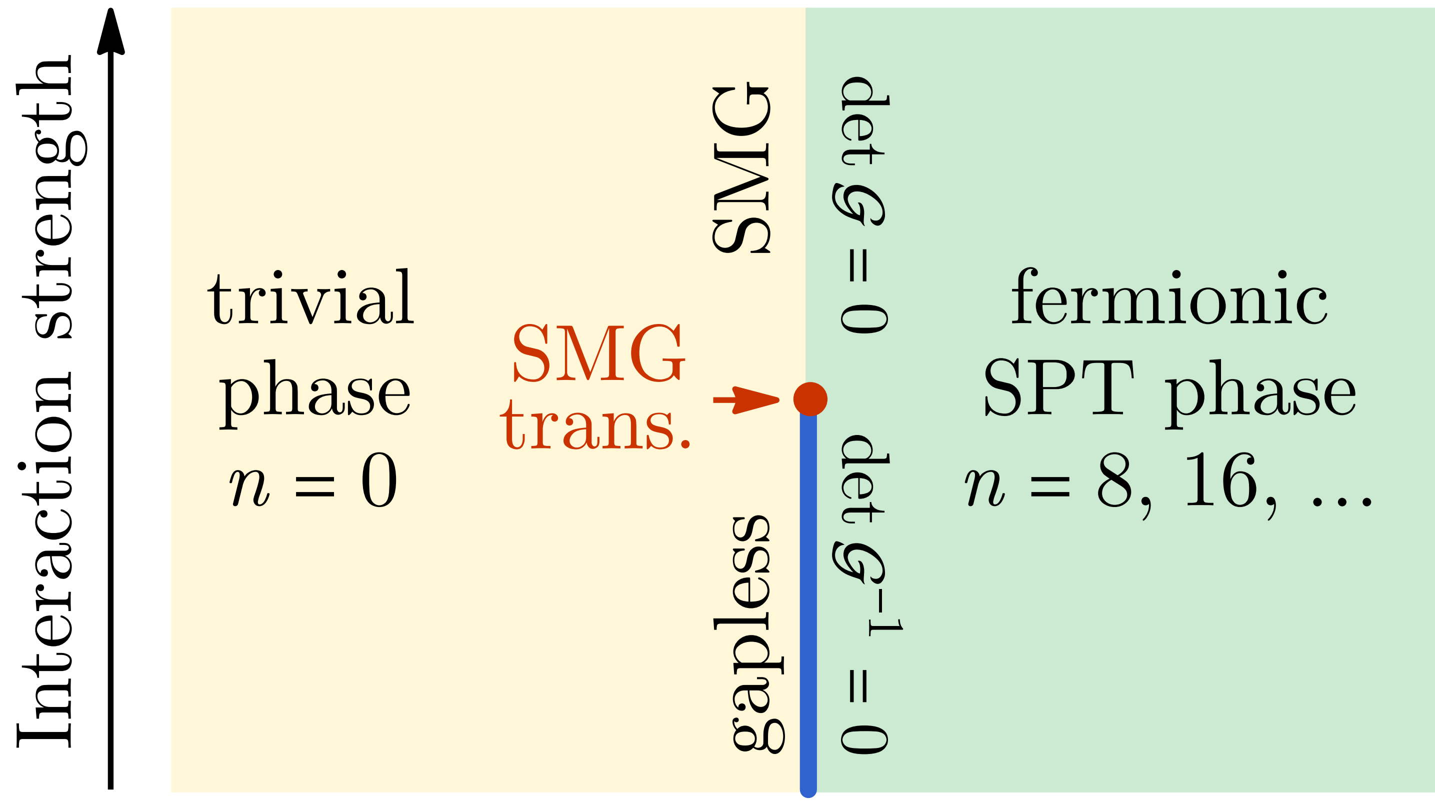
$$n = \int \mathrm{d}^d k \, \mathrm{Tr}\, \mathcal{B}(\mathcal{G}^{-1}\partial\mathcal{G})(\mathcal{G}^{-1}\partial\mathcal{G}) \cdots$$

This is a quantized topological invariant of $\mathcal{G}$ that can not change smoothly.

YZ You, Z Wang, J Oon, C Xu, 1403.4938

Fermion Green's Function Zero

- To see that $\det \mathcal{G} = 0$ must happen in the SMG phase:



$$n = \int \mathrm{d}^d k \, \mathrm{Tr}\, \mathcal{B}(\mathcal{G}^{-1}\partial\mathcal{G})(\mathcal{G}^{-1}\partial\mathcal{G}) \cdots$$

- The index n must change abruptly across the boundary
- This can only happen if the integrand becomes singular, i.e.

$$\det(\mathcal{G}) = 0 \text{ or } \det(\mathcal{G}^{-1}) = 0$$

YZ You, Z Wang, J Oon, C Xu, 1403.4938

Summary

- **Symmetric Mass Generation**:
  a novel mechanism to give fermion a mass without any bilinear condensation, allowing gapping out fermions without breaking symmetry.
- **Conditions**:
  - Kinematics: anomaly cancellation
  - Dynamics: gapping condition (less well understood)
- **Features**:
  - Fermionic deconfined quantum criticality (at the SMG transition): a non-trivial CFT with enlarged fermion scaling dimension
  - Fermion Green's function zero (in the SMG phase)

Summary

- **Symmetric Mass Generation**:
  a novel mechanism to give fermion a mass without any bilinear condensation, allowing gapping out fermions without breaking symmetry.
- **Applications**:
  - Lattice regularization of anomaly-free chiral fermions/gauge theories (e.g. Standard Model or Grand Unified Theories) XG Wen, C Xu, YZ You, BT Yoni, D Tong …
  - New candidate non-SUSY dualities A Karasik, K Onder, D Tong
  - (Potentially) New perspectives on strong CP problem J Wang
  - Classification/construction of interacting SPT states
  - (Potentially) New insights into pseudo-gap physics in high-T_c superconductors

Symmetry Extension

- Key idea: lift the symmetry obstruction by extending the symmetry group G to a larger group $\tilde{G}$, defined by the short exact sequence
$$1 \to K \to \tilde{G} \to G \to 1$$
such that there exists a subgroup $G' \subseteq \tilde{G}$ that
    1. is isomorphic to $G' \cong G$
    2. admits the branching rule
    $$\mathbf{r}_{\psi}^{\tilde{G}} \times_{\mathcal{A}} \mathbf{r}_{\psi}^{\tilde{G}} \to \mathbf{1}^{G'}$$
    under $\tilde{G} \to G'$ breaking (still preserving the symmetry group isomorphically).